

Rings and Coulomb boxes in dissipative environments

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We study a particle on a ring in presence of a dissipative Caldeira-Leggett environment and derive its response to a DC field. We show how this non-equilibrium response is related to a flux averaged equilibrium response. We find, through a 2-loop renormalization group analysis, that a large dissipation parameter η flows to a fixed point $\eta^R = \hbar/2\pi$. We also reexamine the mapping of this problem to that of the Coulomb box and show that the relaxation resistance, of recent interest, is quantized for large η . For finite $\eta > \eta^R$ we find that a certain average of the relaxation resistance is quantized. We propose a Coulomb box experiment to measure a quantized noise.

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I. INTRODUCTION

Two of the most important mesoscopic structures are rings, for the study of persistent currents, and quantum dots or boxes, for the study of charge quantization. Of particular recent interest is the quantization of the relaxation resistance R_q , defined via an AC capacitance of a single electron box (SEB). Following the prediction of Büttiker, Thomas and Prêtre¹ that $R_q = \hbar/2e^2$ for a single mode resistor, a quantum mesoscopic RC circuit has been implemented in a two-dimensional electron gas² and $R_q = \hbar/2e^2$ has been measured. The theory has been recently extended to include Coulomb blockade effects^{3,4} showing that $R_q = \hbar/2e^2$ is valid for small dots and crosses over to $R_q = \hbar/e^2$ for large dots.

In parallel, recent data has observed Aharonov-Bohm oscillations from single electron states in semiconducting rings⁵. Further theoretical works have considered the effects of dissipative environments on a single particle in a ring⁶, in particular studying the renormalization of the mass M^* and its possible relation to dephasing⁶⁻⁹. A related case of a ring coupled by tunneling to an electron lead has also been studied¹⁰.

It is rather remarkable that the ring and box problems are related via the Ambegaokar, Eckern, and Schön (AES) mapping¹¹ where the ring experiences a Caldeira-Leggett (CL)¹² environment. While the exact mapping assumes weak tunneling into the box with many channels, it has been extensively used to describe various tunnel junctions¹³, the Coulomb blockade phenomena in SEB and in the single electron transistor (SET)¹³⁻²¹.

The ring problem is defined by a particle confined to a ring, coupled to a dissipative environment of the Caldeira-Leggett type, and in presence of a field E , generated by a time dependent flux through the ring. This scenario is schematically illustrated in Fig.1. In the present work we address the ring problem by the real time Keldysh method and study it using a 2-loop expansion and renormalization group (RG) reasoning. We find that perturbation theory identifies an unexpected new small parameter $\sin(\frac{\hbar}{2\eta})$ where η is the dissipation parameter on the ring, or the lead-dot coupling in the SEB. We infer that a large η flows to a fixed point η^R with $\hbar/2\eta^R = \pi$. While the thermodynamics of the ring type problem has been much studied, including extensive Monte Carlo studies^{16,19} of M^* , no sign of a finite coupling fixed point has been detected. Our method evaluates the response to a strictly DC electric field E , equivalent to a magnetic flux through the ring that increases linearly with time, hence a non-equilibrium response. We claim that thermodynamic quantities like M^* , that are flux sensitive, decouple from the response to E , a response that

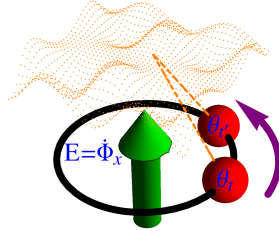


FIG. 1: Artist view of a particle on a ring, coupled to an environment, with a field $E = \dot{\phi}_x$ due to a time dependent flux through the ring. The particle polarizes the environment which in turn modifies the motion of the particle at later times, i.e. an effective non-local interaction.

averages over flux values. This general relation between non-equilibrium and equilibrium responses is given by Eq. (39) below. This relation has been noticed for a model with particle tunneling between a ring and an environment²².

In terms of the SEB, our results extend the previous analysis^{3,4} to the case of many channels N_c , an experimentally realizable scenario²³. We note that for $N_c > 1$ the relaxation resistance for noninteracting electrons¹ becomes $\hbar/(2N_c e^2)$. We find that for strong coupling, $\eta/\hbar \gtrsim 1$ the relaxation resistance is quantized to e^2/\hbar up to an exponentially small correction $\sim e^{-\pi\eta/\hbar}$. For finite η , but still $\eta > \eta^R$ we find that a certain average of the relaxation resistance is quantized (see Eq. 82).

The present work considerably expands our previous letter²⁴. In section II we present the ring and box models, with some exact general properties. In section III we present RG and numerical solutions for the semiclassical case, while section IV presents the perturbation and RG analysis of the full quantum case. The discussion section V summarizes our results, presents a topological interpretation of our fixed point and details a proposed Coulomb box experiment to detect our predicted quantized noise. The Appendices give details of the ring-box mapping and of the various perturbation expansions.

II. THE MODEL AND GENERAL PROPERTIES

A. Semiclassical model

We derive first a Langevin equation for a particle on a ring. Consider the standard Langevin equation for a particle with coordinate x_t in one-dimension of the form

$$R_{t,t'}^{-1} x_{t'} = \xi_t \quad (1)$$

where ξ_t is a Gaussian random force from an environment, where the average on the environment degrees of freedom is

$$\langle \xi_t \xi_{t'} \rangle = B_{t,t'} \quad (2)$$

This relation defines a linear response for either $x_\omega = R_\omega \xi_\omega$ or $\xi(\omega) = R^{-1}(\omega)x(\omega)$, after Fourier transforms, e.g. $R(\omega)$ is the Fourier transform of $R_t = R_{t,0}$. Hence the fluctuation dissipation theorem (FDT) at temperature T can be applied either way, leading to

$$\begin{aligned} K_x(\omega) &= \hbar \coth(\tfrac{1}{2}\beta\hbar\omega) \text{Im}[R_\omega] \\ B_\omega &= \hbar \coth(\tfrac{1}{2}\beta\hbar\omega) \text{Im} \frac{-1}{R_\omega} \end{aligned} \quad (3)$$

where $K_x(\omega)$ is the Fourier transform of $K_x(\tau) = \frac{1}{2}\langle x_t x_{t+\tau} + x_{t+\tau} x_t \rangle$. The simplest choice corresponds to a particle with mass m and a friction coefficient η , so that at temperature $T = 0$

$$\begin{aligned} m\ddot{x}_t + \eta\dot{x}_t &= \xi_t \\ R_0(\omega) &= \frac{-1}{m\omega^2 + i\omega\eta} \quad R_0(t) = \frac{1}{\eta}[1 - e^{-\eta t/m}]\Theta(t) \\ B_\omega &= \hbar\eta|\omega| \quad B_t = \frac{-\hbar\eta}{\pi t^2} \quad (t \neq 0) \end{aligned} \quad (4)$$

where $R_0(t - t')$ is the response in this case. While the mass provides a high frequency cutoff which we denote $\omega_c = \eta/m$, the singularity of $B(t)$ at $t = 0$ implies the need for an additional cutoff. This additional cutoff is a convenience and will be used below in the simulations as well as in the RG derivation. A method for deriving general response functions is based on Kramers Kronig relations³⁰. In the notation of Eq. (2.7) of Ref. [30] we choose $\text{Re}\mu(\omega) = \eta/(1 + \omega^2\tau_0^2)$ so that the response function $R^{-1}(t - t')$, after Fourier, is

$$R_\omega^{-1} = -m\omega^2 - \frac{i\omega\eta}{1 - i\omega\tau_0} \quad (5)$$

which has the remarkable and necessary property that both R_ω and R_ω^{-1} have no poles in the upper half plane. The FDT at $T = 0$ gives

$$B_\omega = \frac{\hbar|\omega|\eta}{1 + \omega^2\tau_0^2} \quad (6)$$

so that $1/\tau_0$ provides a cutoff on the environment frequencies. Hence for $4\tau_0 < m/\eta$, ($\delta \rightarrow +0$)

$$R_t = \Theta(t) \frac{\tau_0}{m} \left[\frac{m}{\eta\tau_0} e^{-\delta t} + \frac{1-\lambda_1}{\lambda_1 x} e^{-\lambda_1 t/\tau_0} - \frac{1-\lambda_2}{\lambda_2 x} e^{-\lambda_2 t/\tau_0} \right]$$

$$\lambda_1 = \frac{1}{2}[1+x], \quad \lambda_2 = \frac{1}{2}[1-x], \quad x = \sqrt{1 - \frac{4\eta\tau_0}{m}} \quad (7)$$

while for $4\tau_0 > m/\eta$ with $x = \sqrt{\frac{4\eta\tau_0}{m}} - 1$

$$R_t = \Theta(t) \frac{1}{\eta} \left\{ e^{-\delta t} - \left[\frac{1-x^2}{2x} \sin(xt/2\tau_0) + \cos(xt/2\tau_0) \right] e^{-t/2\tau_0} \right\} \quad (8)$$

Consider now the two-dimensional system and its projection on a ring, i.e. $\mathbf{x}_t = (\cos \theta_t, \sin \theta_t)$ so that θ_t is the angular position of the particle and the radius is chosen as unity. In cartesian coordinates we define random forces in the x, y directions so that $R_{t-t'}^{-1} \cos \theta_{t'} = -\xi_t^b$, $R_{t-t'}^{-1} \sin \theta_{t'} = \xi_t^a$. The ring potential confines the motion to the azimuthal part, so that only the tangent force $-\xi_t^a \cos \theta_t + \xi_t^b \sin \theta_t$ is allowed, hence

$$-\sin \theta(t) R_{t-t'}^{-1} \cos \theta_{t'} + \cos \theta(t) R_{t-t'}^{-1} \sin \theta_{t'} = \xi_t^a \cos \theta_t + \xi_t^b \sin \theta_t + E \quad (9)$$

where ξ_t^a, ξ_t^b are independent and each having the correlations of Eq. (2). An external tangent electric field E has been added corresponding to a flux through the ring that is increasing linearly with time $\phi_x = Et$. With $R_0(t-t')$ given by Eq. (4) the differential form $R_0^{-1}(t) = m r \partial_t^2 + \eta r \partial_t$, can be used leading to

$$m \ddot{\theta}_t + \eta \dot{\theta}_t = \xi_t^a \cos \theta_t + \xi_t^b \sin \theta_t + E \quad (10)$$

Comparing the terms on the left identifies a cutoff frequency $\omega_c = \eta/m$. At $\omega > \omega_c$ the mass term dominates while at $\omega < \omega_c$ the environment dominates, leading to renormalizations. The nonlinear Langevin's equation (10), including an average on the random forces, is equivalent to a partition function

$$Z = \int \mathcal{D}[\theta, \xi] \delta \left(m \ddot{\theta}_t + \eta \dot{\theta}_t - \xi_t^a \cos \theta_t - \xi_t^b \sin \theta_t - E \right) e^{-\int_\omega [|\xi_\omega^a|^2 + |\xi_\omega^b|^2]/2B_\omega} \quad (11)$$

Introducing the 'quantum' field $\hat{\theta}$ by $\delta(X_t) = \int \mathcal{D}[\hat{\theta}] e^{i\hat{\theta}_t X_t}$, and averaging over the noise field ξ_x, ξ_y results in the semi classical partition function $Z = \int \mathcal{D}[\theta, \hat{\theta}] e^{-S[\theta, \hat{\theta}]}$ where $S[\theta, \hat{\theta}] = S_0 + S_{int}$ is given by

$$S_0 = i \int_{t,t'} \hat{\theta}_t (R_{t,t'})^{-1} \theta_{t'} - iE \int_{t'} \hat{\theta}_{t'} = i \int_\omega R_\omega^{-1} \hat{\theta}_\omega \theta_{-\omega} - iE \int_{t'} \hat{\theta}_{t'}$$

$$S_{int} = \frac{1}{2} \int_{t,t'} \hat{\theta}_t B_{t,t'} \hat{\theta}_t \cos(\theta_t - \theta_{t'}). \quad (12)$$

This has the form of a Keldysh action, with $\theta, \hat{\theta}$ being the classical and quantum fields, respectively. We will see below that this action is the semiclassical $\hbar \rightarrow 0$ limit of the full quantum system.

B. Quantum model

We proceed to define the full quantum problem. The one-dimensional Langevin system^{12,31} has the Keldysh partition $Z = \int \mathcal{D}\hat{x}_t \mathcal{D}x_t e^{-S_K}$ where

$$S_K = i \int_{t,t'} \hat{x}_t R_{t,t'}^{-1} x_{t'} + \frac{1}{2} \int_{t,t'} \hat{x}_t B_{t,t'} \hat{x}_{t'} \quad (13)$$

and \hat{x}_t, x_t are the quantum and classical fields, respectively,

$$x_t = \frac{1}{2}(x_t^+ + x_t^-) \quad \hat{x}_t = \frac{1}{\hbar}(x_t^+ - x_t^-) \quad (14)$$

and x_t^\pm are on the upper and lower Keldysh contour, respectively. On a ring, we use a 2-dimensional vector notation

$$\mathbf{x}_t^+ = [\cos \theta_t^+, \sin \theta_t^+] \quad \mathbf{x}_t^- = [\cos \theta_t^-, \sin \theta_t^-] \quad (15)$$

Defining

$$\theta_t = \frac{1}{2}(\theta_t^+ + \theta_t^-) \quad \hat{\theta}_t = \frac{1}{\hbar}(\theta_t^+ - \theta_t^-) \quad (16)$$

and using trigonometric identities we obtain the quantum action

$$S_K = i \frac{2}{\hbar} \int_{t,t'} R_{t,t'}^{-1} \sin\left(\frac{\hbar}{2} \hat{\theta}_t\right) \cos\left(\frac{\hbar}{2} \hat{\theta}_{t'}\right) \sin(\theta_{t'} - \theta_t) + \frac{2}{\hbar^2} \int_{t,t'} B_{t,t'} \sin\left(\frac{\hbar}{2} \hat{\theta}_t\right) \sin\left(\frac{\hbar}{2} \hat{\theta}_{t'}\right) \cos(\theta_{t'} - \theta_t) \quad (17)$$

We note that the path integral involves continuous θ_t trajectories that can involve n rotations around the ring. Consider the time evolution from an initial wavefunction $\psi(\theta_0, t_0)$ at time t_0 to a final state $\psi(\tilde{\theta}_t, t)$, where both initial and final angles are compact, $0 < \theta_0, \tilde{\theta}_t < 2\pi$,

$$\psi(\tilde{\theta}_t, t) = \int_0^{2\pi} d\theta_0 \sum_n \int_{\theta_0}^{\tilde{\theta}_t + 2\pi n} \mathcal{D}\theta e^{-S(t, t_0)} \psi(\theta_0, t_0) \quad (18)$$

The sum on the integers n expresses that the probability to arrive at a given $\tilde{\theta}_t$ is a sum of probabilities, each with n rotations. The path integral can therefore be written in terms of a decompactified variable $\theta_t = \tilde{\theta}_t + 2\pi n$, i.e. $\sum_n \int_{\theta_0}^{\tilde{\theta}_t + 2\pi n} \mathcal{D}\theta \rightarrow \int_{\theta_0}^{\theta_t} \mathcal{D}\theta$ where now $-\infty < \theta_t < \infty$. This shift does not affect the periodic forms in (17), however it does affect an external electric field E . Consider a time dependent flux $\phi_x(t) = Et$ that contributes to the action a term $\int_{t_i}^{t_f} \phi_x(t) \dot{\theta}_t dt = -E \int_{t_i}^{t_f} \theta_t dt + \phi_x(t_i) \theta_{t_i} - \phi_x(t_f) \theta_{t_f}$. The partial integration is allowed only for the decompactified variable θ_t , i.e. the work done by E is finite for each 2π rotation. The boundary terms are neglected, e.g. one can choose $\phi_x(t_i) = \phi_x(t_f) = 0$ where $t_i, t_f \rightarrow -\infty$ are boundary times on a Keldysh contour; the field E is turned on slowly away from these times.

In the following we will consider a perturbative scheme with a field E and a bare velocity $v = E/\eta$ and θ_t is decomposed to $\theta_t = \delta\theta_t + vt$; (the true velocity is defined below as $v^R(E) = \langle \dot{\theta}_t \rangle$). The velocity v provides a low frequency cutoff eliminating divergence of the perturbative expansion and eventually allows for RG treatment. It will be convenient to use the two-cutoff response Eq. (5) with $R_\omega^{-1} = -m\omega^2 + \delta R_\omega^{-1}$, where $\delta R_\omega^{-1} = \frac{-i\omega\eta}{1-i\omega\tau_0}$, hence

$$\delta R_{t,t'}^{-1} = \partial_{t'} \int_\omega \frac{-\eta}{1-i\omega\tau_0} e^{-i\omega(t-t')} = -\frac{\eta}{\tau_0} \partial_{t'} [e^{-(t-t')/\tau_0} \Theta(t-t')] = \frac{\eta}{\tau_0} e^{-(t-t')/\tau_0} \Theta(t-t') \partial_{t'} \quad (19)$$

The operator identity is satisfied for any function decaying faster than $e^{|t'|/\tau_0}$ at $t' \rightarrow -\infty$. Note,

$$i \int_{t,t'} \hat{\theta}_t \delta R_{t,t'}^{-1} v t' = i \frac{\eta v}{\tau_0} \int_t \hat{\theta}_t \int_{-\infty}^t e^{-(t-t')/\tau_0} dt' = i v \eta \int_t \hat{\theta}_t \quad (20)$$

The mass term with $m\omega^2 \rightarrow \delta(t-t') \partial_t \partial_{t'}$ produces $m \int_t \dot{\theta}_t \dot{\theta}_t = m \int_t \dot{\theta}_t \delta \dot{\theta}_t + m v \int_t \dot{\theta}_t$; the last term with $mv = E/\omega_c$ is neglected relative to the field term $\int_t Et \dot{\theta}_t$. The full action is then

$$\begin{aligned} S_K &= S_0 + S_{int} + S_c \\ S_0 &= i \int_{t,t'} \hat{\theta}_t R_{tt'}^{-1} \theta_{t'} - iE \int_t \hat{\theta}_t = i \int_{t,t'} \hat{\theta}_t R_{tt'}^{-1} \delta\theta_{t'} \\ S_{int} &= \frac{2}{\hbar^2} \int_{t,t'} B_{t,t'} \sin\left(\frac{\hbar}{2} \hat{\theta}_t\right) \sin\left(\frac{\hbar}{2} \hat{\theta}_{t'}\right) \cos(\theta_{t'} - \theta_t) \\ S_c &= i \frac{2}{\hbar} \int_{t,t'} \delta R_{t,t'}^{-1} [\sin\left(\frac{\hbar}{2} \hat{\theta}_t\right) \cos\left(\frac{\hbar}{2} \hat{\theta}_{t'}\right) \sin(\theta_{t'} - \theta_t) - \frac{\hbar}{2} \hat{\theta}_t \theta_{t'}] \end{aligned} \quad (21)$$

The use of a single cutoff Eq. (4) with

$$R_0^{-1}(t, t') = \delta(t-t') [m \partial_t \partial_{t'} + \eta \partial_{t'}] \quad (22)$$

leads to a simpler action. It corresponds to $\tau_0 \rightarrow 0$, hence $\delta R_{t,t'}^{-1} \rightarrow \eta \delta(t-t') \partial_{t'}$,

$$\frac{2}{\hbar} R_0^{-1}(t, t') \sin(\frac{\hbar}{2} \hat{\theta}_t) \cos(\frac{\hbar}{2} \hat{\theta}_{t'}) \sin(\theta_{t'} - \theta_t) = \delta(t-t') [m \dot{\theta}_t \dot{\theta}_t + \frac{\eta}{\hbar} \sin(\hbar \hat{\theta}_t) \dot{\theta}_{t-}] \quad (23)$$

where t^- is infinitesimal below t so that the retarded nature of $R_{t,t'}^{-1}$ is maintained. The action $S_K = S_0 + S_{int} + S_c$ is then

$$\begin{aligned} S_0 &= i \int_{t,t'} \hat{\theta}_t R_0^{-1}(t, t') \delta \theta_{t'} = i \int_t [m \dot{\theta}_t \delta \dot{\theta}_t + \eta \hat{\theta}_t \delta \dot{\theta}_t] = i \int_t [m \dot{\theta}_t \dot{\theta}_t + \eta \hat{\theta}_t \dot{\theta}_t] - iE \int_t \hat{\theta}_t \\ S_{int} &= \frac{2}{\hbar^2} \int_{t,t'} B_{t,t'} \sin(\frac{\hbar}{2} \hat{\theta}_t) \sin(\frac{\hbar}{2} \hat{\theta}_{t'}) \cos(\theta_{t'} - \theta_t) \\ S_c &= \frac{i\eta}{\hbar} \int_t [\sin(\hbar \hat{\theta}_t) \dot{\theta}_{t-} - \hbar \hat{\theta}_t \dot{\theta}_{t-}] \quad \tau_0 \rightarrow 0. \end{aligned} \quad (24)$$

Note that this action reduces to that to the semiclassical case Eq. (12) when $\hbar \rightarrow 0$.

C. Renormalized friction

The renormalized friction $\eta^R(E)$ is defined by the renormalized response $R_{t,t'}^R = i \langle \theta_t \hat{\theta}_{t'} \rangle_E$ and its DC limit:

$$\frac{1}{\eta^R(E)} = \lim_{\omega \rightarrow 0} (-i\omega R_\omega^R) \quad (25)$$

in analogy with the bare form Eq. (4). We show now that the renormalized $\eta^R(E)$ is also the local slope of $\frac{dv^R}{dE}$, where v^R is the E dependent renormalized velocity

$$v^R \equiv \langle \dot{\theta}_t \rangle = \int \mathcal{D}[\theta] \dot{\theta}_t e^{-S_K} \quad (26)$$

Therefore

$$\frac{dv^R}{dE} = i \left\langle \int_{t'} \dot{\theta}_t \hat{\theta}_{t'} \right\rangle = \int_{t'} \frac{d}{dt} R_{t,t'}^R = \int_{t'} \int_\omega (-i\omega) R_\omega^R e^{-i\omega(t-t')} = \lim_{\omega \rightarrow 0} \frac{-i\omega}{-i\eta^R(E)\omega} = \frac{1}{\eta^R(E)} \quad (27)$$

In particular we are interested in the limit $\eta^R = \eta^R(E \rightarrow 0)$.

We show now an alternative procedure for evaluating η^R . Consider the Keldysh partition $Z = \int \mathcal{D}[\theta] e^{-S_K}$ and shift $\hat{\theta}_t \rightarrow \hat{\theta}_t + a_t$. The result must be a_t independent, and choosing the form (23) with $\tau_0 \rightarrow 0$ (the following identity is actually independent of cutoff choices)

$$\begin{aligned} 0 &= \frac{\delta Z}{\delta a_t} \Big|_0 = - \left\langle \frac{\delta(S_0 + S_{int} + S_c)}{\delta \hat{\theta}_t} \right\rangle = -i(\eta v^R - E - \delta E) \\ \delta E &\equiv i \left\langle \frac{\delta(S_{int} + S_c)}{\delta \hat{\theta}_t} \right\rangle \end{aligned} \quad (28)$$

since $-i \langle \frac{\delta S_0}{\delta \hat{\theta}_t} \rangle = -m \langle \ddot{\theta}_t \rangle + \eta \langle \dot{\theta} \rangle - E$ and v^R is time independent, at least for long times.

Taking an E derivative of Eq. (28) and using (27) we obtain

$$\frac{1}{\eta^R(E)} = \frac{1}{\eta} + \frac{1}{\eta^2} \frac{\partial}{\partial v} \delta E \quad (29)$$

We have checked, up to 2nd order terms, that the results of (27) and (29) coincide. The use of (29) is technically easier.

D. Equilibrium correlations

In this section we consider the equilibrium response to a change in flux and derive a relation with the nonequilibrium response to a field.

Consider now the form of $\tilde{K}(\omega)$ as a response to a flux ϕ_x . Linear response to $\delta\mathcal{H}_{ring} = +\hbar\dot{\theta}\delta\phi_x(t)$ is

$$\hbar\langle\dot{\theta}\rangle = - \int_{t'} \tilde{K}_{t,t'} \delta\phi_x(t') \quad (30)$$

This corresponds also to the velocity correlation

$$\tilde{K}_{t,t'} = +i\theta(t-t')\langle[\dot{\theta}_t, \dot{\theta}_{t'}]\rangle \quad (31)$$

We expect that the DC response is positive for small ϕ_x , hence define

$$\tilde{K}(\omega) = -K_0(\phi_x) + i\omega K_1(\phi_x) + O(\omega^2) \quad (32)$$

The response $K_0(\phi_x)$ is the persistent current, i.e. for a static flux one can integrate (30)

$$\langle\dot{\theta}\rangle = \int_0^{\phi_x} K_0(\phi'_x) d\phi'_x \quad (33)$$

The periodicity of the persistent current implies $\int_0^1 K_0(\phi_x) d\phi_x = 0$. The curvature at $\phi_x = 0$ is a well studied object. For general ϕ_x it is defined by a Matsubara imaginary time correlation

$$\frac{1}{\hbar} \frac{\partial^2 F}{\partial \phi_x^2} = (\beta)^{-1} \int_0^\beta \int_0^\beta \langle \dot{\theta}_\tau \dot{\theta}_{\tau'} \rangle^c d\tau d\tau' = K_0(\phi_x) \quad (34)$$

where $K(i\omega_n = 0) = +K_0$ (there is a sign difference in the standard Matsubara notation). An effective mass is defined by $K_0(0) = \hbar/M^*$ so that $M^* = m$ without interactions, while for strong $\eta \gg 1$ coupling $M^* \sim e^{\pi\eta}$ is exponentially large⁶⁻⁹.

To appreciate the role of K_1 consider FDT for the symmetrized correlation at small ω

$$\langle |\dot{\theta}_\omega|^2 \rangle^{sym} = \text{sign}\omega \cdot \text{Im}\tilde{K}_\omega = |\omega| K_1 \quad (35)$$

The diffusion involves the response $\langle |\theta_\omega|^2 \rangle = K_1/|\omega|$, hence for $t \rightarrow \infty$

$$\langle (\theta_t - \theta_0)^2 \rangle = K_1 \int d\omega \frac{1 - \cos\omega t}{\pi|\omega|} = \frac{2K_1}{\pi} \ln(\omega_x t) \quad (36)$$

where ω_x is a characteristic frequency where higher order terms in ω terms set in.

Consider now the linear response to an electric field $\delta\mathcal{H}_{ring} = -E(t)\theta_t$ and use the response $\langle\theta_t\rangle = R_{t,t'}^R E(t')$. The definition (25) implies that the low ω limit has the form $R_\omega^R = \frac{-1}{i\omega\eta^R}$. Since $E = \hbar\dot{\phi}_x$ we expect $\hbar\omega^2 R_\omega^R = \tilde{K}(\omega)$. However, there is a difficulty with the latter relation, if taken literally,

$$\frac{-\hbar\omega^2}{i\omega\eta^R} = ? - K_0(\phi_x) + i\omega K_1(\phi_x) \quad (37)$$

It is also not clear which ϕ_x to use in this relation. To resolve this issue consider the \tilde{K} response with a constant electric field

$$\hbar\langle\dot{\theta}_t\rangle = - \int_{t'} \tilde{K}_{t,t'} \cdot Et' \quad (38)$$

Note first that an additional constant ϕ_x in $\frac{1}{\hbar}Et' + \phi_x$ can be eliminated by redefining the origin of the time t' , hence the persistent current part should be eliminated. More precisely, define $\phi_x(t) = \frac{1}{\hbar}Et$; the $\omega = 0$ component $K_0(\phi_x) = K_0(\frac{1}{\hbar}Et)$ becomes a periodic function, i.e. an AC response with frequency $\omega_E = \frac{2\pi}{\hbar}E$. For $\omega \rightarrow 0$ this persistent current response averages to zero, i.e. $\int_0^1 K_0(\phi_x) d\phi_x = 0$. The same reasoning applies to a ϕ_x average on $K_1(\phi_x)$. Hence for the purpose of evaluating the DC response of (25) we need to average on the flux in (32), hence

$$\lim_{E \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{\tilde{K}(\omega)}{i\omega} = \int_0^1 K_1(\phi_x) d\phi_x = \frac{\hbar}{\eta^R}. \quad (39)$$

The order of limits in (5) signifies that η^R is essentially a non-equilibrium response. The equilibrium - nonequilibrium relation (39) has been noticed in solution of a Boltzmann relaxation equation for particles on a ring, allowing for particle tunneling into an environment²².

The physical picture is that in a DC field the particle rotates around the ring and produces two types of currents. First is the persistent current that oscillates in time as ϕ_x increases and is therefore time averaged to zero; this current is non-dissipative. Second, there is a genuine DC response from the $i\omega K_1$ term, which is dissipative.

E. The Coulomb box

Consider now the Coulomb box system, i.e. a finite region (a "dot") with charging energy E_c coupled by tunneling to a single metallic lead. The Hamiltonian is

$$\mathcal{H} = \sum_k \epsilon_k a_{k,i}^\dagger a_{k,i} + \sum_{\alpha,i} \epsilon_\alpha d_{\alpha,i}^\dagger d_{\alpha,i} + E_c (\hat{N} - N_0)^2 + \sum_{k,\alpha,i} t_{k,\alpha,i} a_{k,i}^\dagger d_{\alpha,i} + h.c. \quad (40)$$

where $i = 1, \dots, N_c$ are channel indices, $d_{\alpha,i}$ are dot electron operators with spectra ϵ_α , $a_{k,i}$ are lead electron operators with spectra ϵ_k , $\hat{N} = \sum_{\alpha,i} d_{\alpha,i}^\dagger d_{\alpha,i}$ is the number operator on the dot, $E_c = e^2/2C_g$ is the charging energy with C_g is the geometric (bare) capacitance, N_0 is the gate voltage in units of $2E_c$. The channel index i is diagonal in the tunneling term, i.e. corresponds to transverse modes that are conserved in tunneling.

Consider the density correlations

$$K_{t,t'} = +i\theta(t-t') \langle [\hat{N}_t, \hat{N}_{t'}] \rangle \quad (41)$$

The AES mapping to the ring problem is reproduced in Appendix A. In particular, N_0 corresponds to $-\phi_x$, $2E_c$ to \hbar^2/m and the relation to the velocity correlation on the ring is

$$\hbar^2 \tilde{K}_{t,t'} = -2E_c \hbar \delta(t-t') + 4E_c^2 K_{t,t'} \quad (42)$$

Using the notation³ $K(\omega) = \hbar \frac{C_0}{e^2} (1 + i\omega C_0 R_q)$, where C_0 is the renormalized capacitance and R_q is the relaxation resistance, we obtain

$$\hbar \tilde{K}(\omega) = -2E_c + 4E_c^2 \frac{C_0}{e^2} (1 + i\omega C_0 R_q) \quad (43)$$

Hence the mapping between the Coulomb box and the ring for the curvature is, using (34)

$$\frac{\hbar^2}{M^*(\phi_x)} = \hbar K_0(\phi_x) = 2E_c \left(1 - \frac{C_0}{C_g}\right) \Rightarrow \frac{m}{M^*(\phi_x)} = 1 - \frac{C_0(N_0)}{C_g} \quad (44)$$

while for the dissipation, using (39)

$$\frac{\hbar}{\eta^R} = \int_0^1 K_1(\phi_x) d\phi_x = \frac{e^2}{\hbar} \int_0^1 \frac{C_0^2(N_0)}{C_g^2} R_q(N_0) dN_0 \quad (45)$$

We note that $\int_0^1 \frac{C_0(N_0)}{C_g} dN_0 = 1$ due to the periodicity of $F(\phi_x)$. An extensive study⁶⁻⁹ of $M^*(0)$ shows that it satisfies $M^*(0) > m$ and that for large η (the bare interaction parameter) $M^*(0)/m \sim e^{\pi\eta} \gg 1$. Hence

$$\frac{C_0}{C_g} = 1 - O(e^{-\pi\eta}) \quad \eta \gtrsim 1 \quad (46)$$

and $C_0 \rightarrow C_g$ for large η .

At this stage we can already propose an interesting experiment for the SEB. By analogy with $E = \hbar \dot{\phi}_x$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 \sim t$. This leads to a DC current into the Coulomb box whose dissipation is the average in Eq. (45). This average is predicted to be quantized, at least for $\eta > \eta^R$, as shown below.

III. SEMICLASSICAL RG AND NUMERICS

A. Perturbations and RG

We study here the action (12) with a perturbation series for correlation functions. Consider first the correlation $C_{t',t} = \langle \theta_{t'} \theta_t \rangle$, which to 1st order is

$$C_{t,t'}^{(1)} = \langle \theta_{t'} \theta_t (-S_{int}) \rangle_{S_0} = \int_{t_1, t_2} B_{t_1, t_2} \cos v(t_1 - t_2) R_{t, t_1} R_{t', t_2} \quad (47)$$

In Fourier space

$$C_\omega^{(1)} = |R_\omega|^2 B_\omega^v \quad (48)$$

where $B_\omega^v = \frac{1}{2} (B_{\omega+v_0} + B_{\omega-v_0})$. Since $C_{t'=t}^{(1)}$ is divergent it is useful to evaluate $\tilde{C}_{t,t'} = \langle [\theta_t - \theta_{t'}]^2 \rangle$, which to 1st order is, with $\tau = t - t'$ ($\tau \gg 1/\omega_c$),

$$\tilde{C}_\tau = \int_\omega B_\omega^v |R_\omega|^2 (1 - \cos \omega \tau) \approx \frac{2\hbar}{\pi \eta} \begin{cases} \log(\frac{\eta \tau}{m}) & \tau < \frac{1}{v} \\ \frac{1}{2} \pi v \tau & \frac{1}{v} < \tau \end{cases}. \quad (49)$$

For $E = 0$ the angular position diffuses logarithmically, while for $E \neq 0$ the long time fluctuation is linear in time.

Consider next the response function to 2nd order in S_{int} ,

$$R_{t,t'}^R = i \langle \hat{\theta}_{t'} \theta_t \rangle = R_{t,t'} + R_{t,t'}^{(1)} + R_{t,t'}^{(2)} = R_{t,t'} + i \langle \hat{\theta}_{t'} \theta_t (-S_{int} + \frac{1}{2} S_{int}^2) \rangle_{S_0} \quad (50)$$

Note that the disconnected terms in the perturbation $\langle S_{int}^n \rangle_{S_0}$ vanish for any order n , due to the normalization $Z = 1$. The first order response function is

$$R_{t,t'}^{(1)} = -i \frac{1}{2} \int_{t_1, t_2} B_{t_1, t_2} \langle \hat{\theta}_{t_1} \hat{\theta}_{t_2} \cos(\theta_{t_1} - \theta_{t_2}) \hat{\theta}_{t'} \theta_t \rangle_{S_0} \quad (51)$$

The result in frequency variable is (see Appendix B)

$$R_\omega^{(1)} = R_\omega^2 \int_{\omega_1} R_{\omega_1} [B_{\omega_1}^v - B_{\omega-\omega_1}^v] = R_\omega^2 \int_t R_t B_t \cos v_0 t (e^{i\omega t} - 1) \quad (52)$$

We note that for $v = 0$ FDT is maintained, to this order, $C_\omega^{(1)}|_{v=0} = \text{Im} R_\omega \hbar \text{sign}(\omega)$.

The renormalized η to first order is then

$$\begin{aligned} \frac{1}{\eta_1^R} &= \lim_{\omega \rightarrow 0} (-i\omega) R_\omega^{(1)} = \lim_{\omega \rightarrow 0} \frac{-i\omega}{(-i\omega)^2 \eta^2} \int_t R_t B_t \cos vt (i\omega t) \\ &= \frac{1}{2\eta^2} \log(1 + \omega_c^2/v^2) = -\frac{\log v/\omega_c}{\eta^2} + \mathcal{O}(v) \end{aligned} \quad (53)$$

Considering next the 2nd order in (50) we obtain (see Appendix B)

$$\begin{aligned} R_\omega^{(2)} &= R_\omega^2 \left(-\frac{1}{2} \int_t R_t B_t \cos v_0 t (e^{i\omega t} - 1) \tilde{C}_t^{(1)} + \int_t R_t^{(1)} B_t \cos v_0 t (e^{i\omega t} - 1) + \right. \\ &\quad \left. R_\omega \left[\int_t R_t B_t \cos v_0 t (e^{i\omega t} - 1) \right]^2 - \int_{t_1, t_2} R_{t_1} B_{t_1} B_{t_2} \sin v_0 t_1 \sin v_0 t_2 (1 - e^{i\omega t_1}) t_1 \right) \end{aligned} \quad (54)$$

Denoting the contribution of the last term in (54) as $\delta(\frac{1}{\eta_2^R})$ we obtain for the renormalized dissipation to 2nd order

$$\frac{1}{\eta_2^R} = \frac{1}{\eta} - \frac{\log v}{\eta^2} + \frac{\log^2 v - \log v}{\eta^3} + \delta\left(\frac{1}{\eta_2^R}\right) \quad (55)$$

The contribution of the last term is peculiar and depends on the order of limits taken. We define a nonequilibrium limit where η^R is evaluated for a strictly DC field, i.e. $\omega \rightarrow 0$ is taken first, and then a logarithmically divergent $E \neq 0$ term is obtained, i.e.

$$\begin{aligned} \delta\left(\frac{1}{\eta_2^R}\right) &= \frac{1}{\eta^2} \lim_{v \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{1}{i\omega} \int_{t_1, t_2} R_{t_1} B_{t_1} B_{t_2} \sin vt_1 \sin vt_2 (1 - e^{i\omega t_1}) t_1 = \\ &= -\frac{1}{\eta^3} \lim_{v \rightarrow 0} \int_{t_1} R_{t_1} B_{t_1} \sin v_0 t_1 t_1^2 \int_{t_2} R_{t_2} B_{t_2} \sin vt_2 = \lim_{v \rightarrow 0} \frac{1}{\eta^3} \int^\infty \sin(vt_1) \times \int^\infty \sin(vt_2)/t_2^2 = \\ &= \lim_{v \rightarrow 0} \frac{1}{\eta^3} \frac{1}{v} \times v \log v + \mathcal{O}(v) = \frac{1}{\eta^3} \log v \end{aligned} \quad (56)$$

Considering next the alternative equilibrium order of limits, i.e. first $E \rightarrow 0$, we obtain

$$\lim_{\omega \rightarrow 0} \lim_{v \rightarrow 0} \sin(vt_1) \sin(vt_2) = 0 \quad (57)$$

hence $\delta(\frac{1}{\eta_2^R}) = 0$. The renormalized η to second order is then

$$\frac{1}{\eta_2^R} = \frac{1}{\eta} - \frac{\log v}{\eta^2} + \frac{\log^2 v + b_0 \log v}{\eta^3} \quad (58)$$

where b_0 depend on the order of limits, the nonequilibrium case has $b_0 = 0$, while the equilibrium one has $b_0 = -1$. The latter case is in fact the known equilibrium result¹⁵. The distinction between the two limits will become more pronounced in the full quantum treatment.

B. Numerical solution of the Langevin Equation

We solve the nonlinear Langevin equation numerically. The time is discretized to $t = T/N \times (1, 2, \dots, N)$, with T the total time span of system. The noise term ξ_t^i is generated numerically using a discrete Fourier transform of $\xi_\omega^i = \sqrt{B_\omega T} \mathcal{R}^i$ where \mathcal{R}^i is a unit white Gaussian noise. The correlation function linearity requires introducing a high frequency cutoff τ_0 . We choose the cutoff to be in Lorentzian form $B_\omega = \hbar\eta|\omega|/[1 + \omega^2\tau_0^2]$, in the following section we explain the importance of this choice.

We solve the equation in iterative procedure. Using the convolution form

$$\theta_t = \int_{t'} R_{t, t'} [\xi_{t'}^x \cos \theta_{t'} - \xi_{t'}^y \sin \theta_{t'} - E] \quad (59)$$

starting with an arbitrary configuration of $\theta_t^{(0)}$ we calculate the right hand side (RHS) of (59) to find a new $\theta_t^{(1)}$. We repeat the procedure n times until the expression is saturated when $\theta_t^{(n)} = \theta_t^{(n+1)}$. This procedure is improved if instead of taking the convolution result as the next order θ_t we use some mixing of that result and of the previous θ_t configuration in the form $\theta_t^{(m)} = (1 - \beta)\theta_t^{(m-1)} + \beta \times \text{RHS}$ where β is mixing parameter. Typically n would be in order of 10^5 and $\beta = 0.1$.

With this choice the Langevin equation takes the following form

$$\begin{aligned} m\ddot{\theta}_t &= \xi_t^x \cos \theta_t + \xi_t^y \sin \theta_t + E + \Delta_t \\ \Delta_t &= \frac{\eta}{\tau_0^2} \int_{-\infty}^t \sin[\theta_t - \theta_{t'}] e^{-(t-t')/\tau_0} dt', \end{aligned} \quad (60)$$

where Δ_t is a correction term defined by δR_ω^{-1} in the response function Eq. (19) as $\int_{t'} \delta R_{t, t'}^{-1} [\xi_{t'}^x \cos \theta_{t'} + \xi_{t'}^y \sin \theta_{t'} + E] = -\int_\omega m\omega^2 \Delta_\omega$

In the numerical system we have now four time scales, two numerical time scales, i.e. the time segment $\Delta\tau = \bar{T}/N$ and the time span \bar{T} , as well as the two physical high frequency cutoffs, $1/\tau_0$ for the noise and ω_c the mass cutoff. The region of interest corresponds to velocity $v^R = \langle \dot{\theta}_t \rangle$ between the time scales $\Delta\tau \ll \tau_0 < 1/\omega_c \ll 1/v^R \sim 1/v < \bar{T}$. The inequality $\tau_0 < 1/\omega_c$ is useful since we compare the numerical result to an asymptotic result in which ω_c rather than $1/\tau_0$ is the high frequency cutoff.

With the result for θ_t we can find the renormalized $1/\eta^R = dv^R/dE$ with $v^R = \langle \dot{\theta}_t \rangle$ where the average $\langle \dots \rangle$ reflects an average on both the time domain $t > 1/\omega_c$ and on numerous realizations of the noise.

In the left panel of Fig.2 our numerical solution for the Langevin equation is shown, including a fit to the second order with $b_0 = 0$. On the right panel the 1st order is subtracted with either the nonequilibrium $b_0 = 0$ or the equilibrium $b_0 = -1$. The first is in fact a better fit to the numerical data. When $1/v$ approaches the simulation time span \bar{T} the numerics become unreliable, as the particle cannot complete even one revolution in time \bar{T} ; a plateau is then observed at low E .

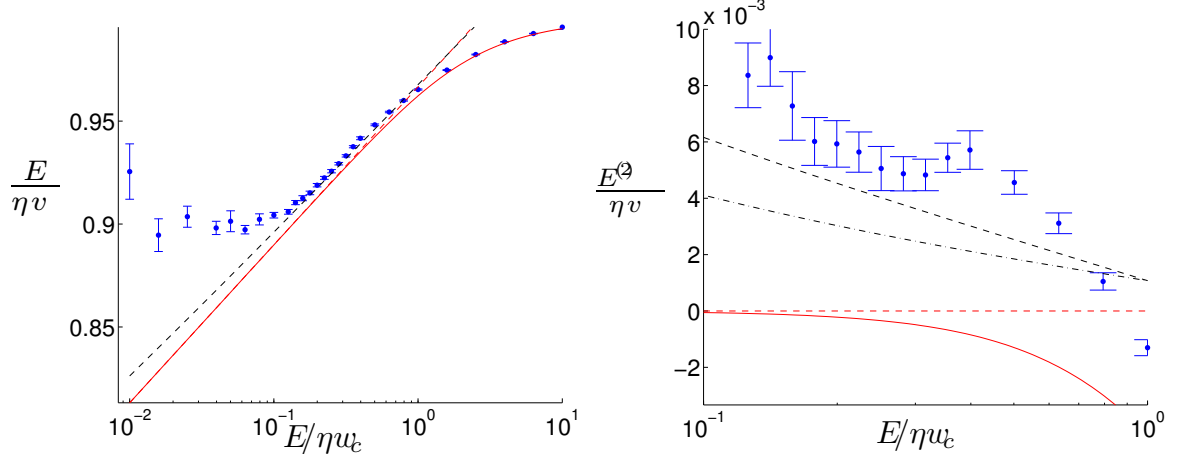


FIG. 2: **Left panel:** Velocity-field relation for Eq. (60) with $\eta = 30\hbar/\pi$, $\omega_c = 100/\tau_0$ and $\tau_0 = 20\Delta\tau$. Here $N = 2^{15}$, $\Delta\tau = 1/20$. The circles are numerical data, the full red line is a 1st order perturbation in $1/\eta$, the dashed lower red line is its logarithmic expansion for large $\ln v/\omega_c$ and the dashed upper (black) line includes the 2nd order logarithmic term, corresponding to Eq. (58) for $b_0 = 0$. Note that the data is not reliable for $E/\eta\omega_c \lesssim 1/(\Delta\tau N\omega_c) \sim 0.06$.

Right panel: The same data and line types after subtracting the 1st order terms, i.e. $\frac{E^2}{\eta v} = \frac{E}{\eta v} - 1 - \frac{\hbar}{\pi\eta}(\ln \frac{v}{\omega_c} - 1)$. An additional dash-dotted line corresponds to $b_0 = -1$, which is a worse fit to the data than $b_0 = 0$ (dashed upper line). Note that the numerical data displays E/v rather than dE/dv , hence Eq. (53) acquires a -1 term.

With the numerical results for θ_τ we can also generate the correlation function $\tilde{C}_\tau = \langle [\theta_\tau - \theta_0]^2 \rangle$, the first order perturbation for this correlation function is given in Eq. (49). In Fig. 3 we plot this correlation function as a function of the time separation τ for the same parameters as in Fig.2, with and without a finite field. The data is fairly close to the 1st order result (49) for not too long times, i.e. for zero field the correlation has a subdiffusion logarithmic behavior while for finite force the correlation has a diffusion ($\sim \tau$) behavior.

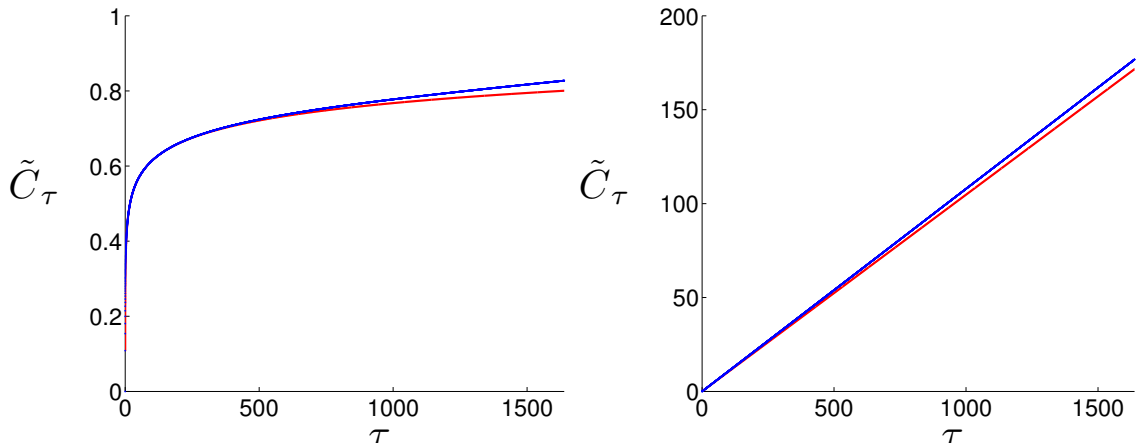


FIG. 3: **Left panel:** The correlation function \tilde{C}_τ as a function of time (Blue) and the asymptotic results of Eq. (49) (red) for $E = 0$. **Right panel:** The correlation function as a function of time (Blue) and the asymptotic results of Eq. (49) for $E/\eta = 1$ and $\tau_0 = 1$.

IV. QUANTUM RG

A. Perturbations from S_{int}

Consider now the definition η^R in Eqs. (28,29)

$$\begin{aligned}
 -i\delta E^{(1)} &= \left\langle \frac{\delta S_{int}}{\delta \hat{\theta}_t} \right\rangle_0 = \frac{2}{\hbar} \int_{t'} B_{t,t'} \left\langle \cos\left(\frac{\hbar}{2}\hat{\theta}_t\right) \sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right) \cos(vt - vt' + \delta\theta_t - \delta\theta_{t'}) \right\rangle_0 = \\
 &= \frac{2}{\hbar} \int_{t'} B_{t,t'} \sum_{\sigma, \sigma', \mu = \pm} \frac{\sigma'}{8i} \left\langle e^{\frac{1}{2}i\hbar\sigma\hat{\theta}_t + \frac{1}{2}i\hbar\sigma'\hat{\theta}_{t'} + i\mu(vt - vt' + \delta\theta_t - \delta\theta_{t'})} \right\rangle_0 = \\
 &= \frac{2}{\hbar} \int_{t'} B_{t,t'} \sum_{\sigma, \sigma', \mu = \pm} \frac{\sigma'}{8i} e^{-\frac{1}{2}\mu\hbar(\sigma i R_{t't} - \sigma' i R_{tt'}) + i\mu(vt - vt')} \quad (61)
 \end{aligned}$$

For $t < t'$ the term $\sigma' R_{tt'} = 0$ and then $\sum \sigma' = 0$. The result is then finite only for $t > t'$; defining $\mu' = \mu\sigma'$,

$$= \frac{2}{\hbar} \int_{t'} B_{t,t'} \sum_{\sigma', \mu' = \pm} \frac{\sigma'}{4i} e^{i\sigma'\mu'(vt - vt') + \frac{1}{2}i\hbar\mu'R_{tt'}} = i \frac{2}{\hbar} \int_{t'} B_{t,t'} \sin v(t - t') \sin(\frac{1}{2}\hbar R_{tt'}) \quad (62)$$

Hence the force correction is

$$\delta E^{(1)} = -\frac{2}{\hbar} \int_{\tau} B_{\tau} \sin(\frac{1}{2}\hbar R_{\tau}) \sin(v\tau) \quad (63)$$

so that using Eq. (29) and performing the calculation of the integrals with arbitrary cutoffs τ_0 and $\omega_c^{-1} = m/\eta$ one obtains:

$$\frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi\eta} \left[\sin\left(\frac{\hbar}{2\eta}\right) \ln(v/\omega_c) + C + O(1/v) \right] \quad (64)$$

where the constant C depends on τ_0 and ω_c . Although we will not need it below, its detailed form is given in the Appendix C in the limit $\tau_0 = 0$.

Consider next 2nd order in S_{int} ,

$$\begin{aligned}
 i\delta E^{(2)} &= \frac{1}{2} \left\langle \frac{\delta}{\delta \hat{\theta}_{t_1}} S_{int}^2 \right\rangle = \frac{1}{2} 4 \left(\frac{2}{\hbar^2}\right)^2 \frac{\hbar i}{2 \cdot 2^6} \sum_{\epsilon_i, \sigma, \sigma' = \pm} \epsilon_2 \epsilon_3 \epsilon_4 \int_{t_2, t_3, t_4} B_{t_1, t_2} B_{t_3, t_4} e^{i\sigma v(t_1 - t_2) + i\sigma' v(t_3 - t_4)} \\
 &\times \left\langle e^{\frac{1}{2}i(\epsilon_1 \hat{\theta}_{t_1} + \epsilon_2 \hat{\theta}_{t_2} + \epsilon_3 \hat{\theta}_{t_3} + \epsilon_4 \hat{\theta}_{t_4}) + i\sigma(\theta_{t_1} - \theta_{t_2}) + i\sigma'(\theta_{t_3} - \theta_{t_4})} \right\rangle_0 \quad (65)
 \end{aligned}$$

Note that $\delta/\delta \hat{\theta}_{t_1}$ can be applied also at either t_2, t_3, t_4 and all these terms are identical since $\sin(\frac{1}{2}\hbar \hat{\theta}_{t_i})$ appears in the same form for all t_i , hence a factor 4. Now change all $\epsilon_i, \sigma, \sigma' \rightarrow -(\epsilon_i, \sigma, \sigma')$ and define $\sigma' = \sigma\mu$ to obtain

$$\begin{aligned}
 i\delta E^{(2)} &= \frac{i^2}{16\hbar^3} \sum_{\epsilon_i, \sigma, \sigma' = \pm} \epsilon_2 \epsilon_3 \epsilon_4 \sigma \int_{t_2, t_3, t_4} B_{t_1, t_2} B_{t_3, t_4} \sin[v(t_1 - t_2) + \mu v(t_3 - t_4)] \\
 &\times \exp\left\{-\frac{1}{2}\hbar \left\langle \sigma(\epsilon_1 \hat{\theta}_{t_1} + \epsilon_2 \hat{\theta}_{t_2} + \epsilon_3 \hat{\theta}_{t_3} + \epsilon_4 \hat{\theta}_{t_4}) [\theta_{t_1} - \theta_{t_2} + i\mu(\theta_{t_3} - \theta_{t_4})] \right\rangle_0 \right\} \\
 &\frac{-1}{8\hbar^3} \sum_{\epsilon_i, \mu} \epsilon_2 \epsilon_3 \epsilon_4 \int_{t_2, t_3, t_4} B_{t_1, t_2} B_{t_3, t_4} A_2 \sin[v(t_1 - t_2 + \mu v(t_3 - t_4))] \quad (66)
 \end{aligned}$$

where

$$\begin{aligned}
 A_2 &= \exp\left\{\frac{1}{2}i\hbar\epsilon_1(-R_{t_2, t_1} + \mu R_{t_3, t_1} - \mu R_{t_4, t_1})\right\} \times \\
 &\exp\left\{\frac{1}{2}i\hbar\epsilon_2(R_{t_1, t_2} + \mu R_{t_3, t_2} - \mu R_{t_4, t_2})\right\} \times \\
 &\exp\left\{\frac{1}{2}i\hbar\epsilon_3(R_{t_1, t_3} - R_{t_2, t_3} - \mu R_{t_4, t_3})\right\} \times \\
 &\exp\left\{\frac{1}{2}i\hbar\epsilon_4(R_{t_1, t_4} - R_{t_2, t_4} + \mu R_{t_3, t_4})\right\}. \quad (67)
 \end{aligned}$$

Note that in A_2 if t_2 is the maximal time then its second factor =1 and $\sum_{\epsilon_2} \epsilon_2 = 0$. similarly, if t_3 (or t_4) is the maximal time, the the 3rd (or 4th) factor =1 and $\sum_{\epsilon_3} \epsilon_2 = 0$ (or $\sum_{\epsilon_4} \epsilon_2 = 0$). Therefore t_1 must be the maximal time and the 1st factor =1. The result is symmetric in $t_3 \leftrightarrow t_4$, so choose $t_3 > t_4$, with factor 2. Hence 3 time orderings, denoted by A,B,C, $\delta E^{(2)} = \delta E_A + \delta E_B + \delta E_C$,

$$\begin{aligned}
\delta E_A &= \frac{4}{\hbar^3} \sum_{\mu} \int_{t_1 > t_2 > t_3 > t_4} \sin(\frac{1}{2}\hbar R_{t_1,t_2}) \sin[\frac{1}{2}\hbar(R_{t_1,t_3} - R_{t_2,t_3})] \sin[\frac{1}{2}\hbar(R_{t_1,t_4} - R_{t_2,t_4} + \mu R_{t_3,t_4})] \\
&\quad \times B_{t_1,t_2} B_{t_3,t_4} \sin[v(t_1 - t_2) + \mu v(t_3 - t_4)] \\
\delta E_B &= \frac{4}{\hbar^3} \sum_{\mu} \int_{t_1 > t_3 > t_2 > t_4} \sin[\frac{1}{2}\hbar(R_{t_1,t_2} + \mu R_{t_3,t_2})] \sin(\frac{1}{2}\hbar R_{t_1,t_3}) \sin[\frac{1}{2}\hbar(R_{t_1,t_4} - R_{t_2,t_4} + \mu R_{t_3,t_4})] \\
&\quad \times B_{t_1,t_2} B_{t_3,t_4} \sin[v(t_1 - t_2) + \mu v(t_3 - t_4)] \\
\delta E_C &= \frac{4}{\hbar^3} \sum_{\mu} \int_{t_1 > t_3 > t_4 > t_2} \sin[\frac{1}{2}\hbar(R_{t_1,t_2} + \mu R_{t_3,t_2} - \mu R_{t_4,t_2})] \sin(\frac{1}{2}\hbar R_{t_1,t_3}) \sin[\frac{1}{2}\hbar(R_{t_1,t_4} + \mu R_{t_3,t_4})] \\
&\quad \times B_{t_1,t_2} B_{t_3,t_4} \sin[v(t_1 - t_2) + \mu v(t_3 - t_4)]
\end{aligned} \tag{68}$$

B and C terms can be time ordered as A by $t_2 \leftrightarrow t_3$ in B and $t_2 \rightarrow t_4, t_4 \rightarrow t_3, t_3 \leftrightarrow t_2$ in C. In terms of the $\mu = \pm$ components,

$$\begin{aligned}
\delta E_A^+ + \delta E_C^- &= \frac{4}{\hbar^3} \int_A \sin(\frac{1}{2}\hbar R_{t_1,t_2}) \sin[\frac{1}{2}\hbar(R_{t_1,t_3} - R_{t_2,t_3})] \sin[\frac{1}{2}\hbar(R_{t_1,t_4} - R_{t_2,t_4} + R_{t_3,t_4})] \\
&\quad \times [B_{t_1,t_2} B_{t_3,t_4} + B_{t_1,t_4} B_{t_2,t_3}] \sin[v(t_1 - t_2 + t_3 - t_4)] \\
\delta E_A^- + \delta E_B^- &= \frac{4}{\hbar^3} \int_A \sin(\frac{1}{2}\hbar R_{t_1,t_2}) \sin[\frac{1}{2}\hbar(R_{t_1,t_3} - R_{t_2,t_3})] \sin[\frac{1}{2}\hbar(R_{t_1,t_4} - R_{t_2,t_4} - R_{t_3,t_4})] \\
&\quad \times [B_{t_1,t_2} B_{t_3,t_4} + B_{t_1,t_3} B_{t_2,t_4}] \sin[v(t_1 - t_2 + t_4 - t_3)] \\
\delta E_B^+ + \delta E_C^+ &= \frac{4}{\hbar^3} \int_A \sin(\frac{1}{2}\hbar R_{t_1,t_2}) \sin[\frac{1}{2}\hbar(R_{t_1,t_3} + R_{t_2,t_3})] \sin[\frac{1}{2}\hbar(R_{t_1,t_4} - R_{t_3,t_4} + R_{t_2,t_4})] \\
&\quad \times [B_{t_1,t_3} B_{t_2,t_4} + B_{t_1,t_4} B_{t_2,t_3}] \sin[v(t_1 - t_3 + t_2 - t_4)]
\end{aligned} \tag{69}$$

In general we have two cutoffs $m/\eta, \tau_0$ in Eq. (7) and we define $\tau_1(m/\eta, \tau_0)$ as the cutoff time for the response R_t , Eq. (7). For the purpose of identifying the leading $\ln^2 v$ term we take a formal limit such that this cutoff time is $\tau_1 \rightarrow 0$. We will eventually restore physical cutoffs corresponding to $m/\eta, \tau_0$ in R_t . The only cutoff for now is τ_0 in $B(\omega)$, Eq. (6). In this limit $R_t \rightarrow \frac{1}{\eta} \Theta(t) e^{-\delta t}$ where $\delta \rightarrow +0$ to ensure the retarded nature (poles of $1/(\omega + i\delta)$). The significant virtue of this limit is that the 1st two equations of (69) vanish since $R_{t_1,t_3} - R_{t_2,t_3} \rightarrow 0$, leaving just the last form. The evaluation of $\delta E^{(2)}$ in this limit is straightforward (Appendix D), leading to

$$\delta E^{(2)} = \frac{4\eta^2}{\pi^2 \hbar} \sin^2(\frac{\hbar}{2\eta}) \sin(\frac{\hbar}{\eta}) \cdot v \ln(v\tau_0) [\ln(v\tau_0) + 1] \tag{70}$$

Hence from (29)

$$\frac{1}{\eta^{R(2)}} = \frac{4}{\pi^2 \hbar} \sin^2(\frac{\hbar}{2\eta}) \sin(\frac{\hbar}{\eta}) \cdot [\ln^2(v\tau_0) + 3 \ln(v\tau_0) + 1] \tag{71}$$

So far $\delta E^{(2)}$ is calculated in a formal limit $\tau_1 \rightarrow 0$. Instead of attempting the hard calculation of (69) for $\tau_1 \neq 0$, we proceed by asserting that for any τ_0, τ_1 the leading singularity as $v \rightarrow 0$ is a $\ln^2 v$ term, as expected for a 2-loop calculation. This term must involve an η dependent function $f_\eta(\tau_0, \tau_1)$ that has dimensions of time. Fixing the coefficient of $\ln^2[v f_\eta(\tau_0, \tau_1)]$ as in Eq. (71), we have $f_\eta(\tau_0, 0) = \tau_0$ while for $\tau_0 \rightarrow 0$, when $\tau_1 \rightarrow m/\eta = 1/\omega_c$ we must have the form $f_\eta(0, \tau_1) = b(\eta)\tau_1 = b(\eta)/\omega_c$. The 2-loop correction Eq. (71) becomes at $\tau_0 = 0$

$$\frac{1}{\eta^{R(2)}} = \frac{4}{\pi^2 \hbar} \sin^2(\frac{\hbar}{2\eta}) \sin(\frac{\hbar}{\eta}) \cdot \ln^2[\frac{v}{\omega_c} b(\eta)] + O(\ln v) \tag{72}$$

The renormalized friction has therefore the form

$$\frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi \eta} \sin(\frac{\hbar}{2\eta}) \ln[\frac{v}{\omega_c}] + \frac{4}{\pi^2 \hbar} \sin^2(\frac{\hbar}{2\eta}) \sin(\frac{\hbar}{\eta}) \{ \ln^2[\frac{v}{\omega_c}] + b_0(\eta) \ln[\frac{v}{\omega_c}] \} \tag{73}$$

We have thus identified the coefficient of the \ln^2 term. We expect that the \ln coefficient $b_0(\eta)$ is a smooth function of η and in particular is not singular at the zeroes of $\sin \frac{\hbar}{2\eta}$. For $\hbar/\eta \ll 1$ we have found $b_0 = 0$, Eq. (58). Hence, if $b_0(\eta)$ were singular and periodic, e.g. $b_0(\eta) \sim 1/\sin \frac{\hbar}{2\eta}$, it would alter the 1st order result at $\hbar/\eta \ll 1$, an incorrect situation. Furthermore a singular $b_0(\eta)$ is inconsistent with the RG analysis, as based on the \ln^2 coefficient, in subsection C.

We note that in the semiclassical limit the perturbation expansion is in $R^{2n-1}B^n/\eta^2 \sim 1/\eta^{n+1}$ for large η ; in the quantum case the R^{2n-1} factors become periodic functions. The main conclusion is that there is a new small parameter in the perturbation series, $\sin(\frac{\hbar}{2\eta})$.

B. Perturbations from S_c

Here we consider the S_c interaction in Eq. (21). The S_c terms are

$$\langle \hat{\theta}_{t'} \theta_t S_c \rangle = \langle \hat{\theta}_{t'} \theta_t S_c^2 \rangle = 0 \quad (74)$$

However, the mixed term and the corresponding correction to $1/\eta$ are

$$\begin{aligned} \delta R_{t,t'}^m &= i \langle \hat{\theta}_{t'} \theta_t S_c S_{int} \rangle \\ \Rightarrow \frac{1}{\eta^m} &= \frac{2}{\pi \hbar} \left[\sin \frac{\hbar}{2\eta} \left(\sin \frac{\hbar}{\eta} - \frac{\hbar}{\eta} \right) + \frac{\hbar}{2\eta} \cos \frac{\hbar}{2\eta} \left(\sin \frac{\hbar}{\eta} - \frac{\hbar}{\eta} \right) \right] \ln(v\tau_1) \end{aligned} \quad (75)$$

which does not vanish at $\sin \frac{\hbar}{2\eta} = 0$. Note, however that this term is $\sim \hbar^3$, i.e. a 3-loop term. Furthermore, other response functions do show such zeroes. E.g. for the $\bar{R}_{t,t'}$ correlation (Eq. (77) below) we have $\langle \theta_t \sin \frac{\hbar}{2} \hat{\theta}_{t'} S_c \rangle = 0$ to 1st order, while in 2nd order

$$\begin{aligned} \delta \bar{R}_{t,t'}^m &= \frac{2i}{\hbar} \langle \theta_t \sin \frac{\hbar}{2} \hat{\theta}_{t'} S_c S_{int} \rangle \\ \Rightarrow \frac{1}{\bar{\eta}^m} &= \frac{2}{\pi \hbar} \sin \frac{\hbar}{\eta} \left(\sin \frac{\hbar}{\eta} - \frac{\hbar}{\eta} \right) \ln(v\tau_1) \end{aligned} \quad (76)$$

We note that there are many other operators that have vanishing perturbations at $\sin \frac{\hbar}{2\eta} = 0$ to 2nd order in S_{int} , S_c , e.g. the dissipation term in Eq. (9) $\langle \theta_t \sin(\hbar \hat{\theta}_{t'}) \rangle$, or the response to an AC field with frequency v $\langle \theta_t \cos \delta \theta_{t'} \sin \frac{\hbar}{2} \hat{\theta}_{t'} \rangle$.

C. RG analysis

We note that in (73) $g = \frac{2}{\pi} \sin(\frac{\hbar}{2\eta})$ acts as an unexpected small parameter for the expansion, since all divergences vanish when $g = 0$. It raises the interesting possibility that $g = 0$ be viewed as a RG fixed point. For that we need to find a renormalized coupling which obeys multiplicative RG, the simplest choice being $g_R = \frac{2}{\pi} \sin(\frac{\hbar}{2\eta^R(E)})$. The question is then whether the β -function $\beta = -E \partial_E g_R$ can be written only in terms of g_R . Although the non-periodic $1/\eta$ factor in (73) appears at first problematic, we propose that resummation from higher loops, which allows for higher order terms $O(\frac{1}{\eta^4})$ changes the 1-loop term in (73) by $\frac{\hbar}{2\eta} \rightarrow \sin(\frac{\hbar}{2\eta})$.

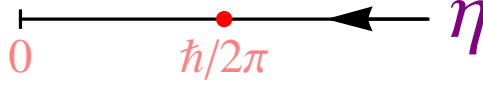
To further motivate this proposal we consider the response

$$\bar{R}_{t,t'} = i \frac{2}{\hbar} \left\langle \theta_t \sin\left(\frac{\hbar}{2} \hat{\theta}_{t'}\right) \right\rangle. \quad (77)$$

Physically, $e^{\pm i \frac{\hbar}{2} \hat{\theta}_{t'}}$ corresponds to an electric field pulse $\delta E(t) = \pm \frac{\hbar}{2} \delta(t - t')$ or equivalently a rapid change of flux by $\pm \frac{1}{2}$, therefore $\bar{R}_{t,t'}$ corresponds to the difference in response to these two flux pulses. Defining the dissipation parameter $\bar{\eta}^R$ for $\bar{R}_{t,t'}$ as in Eq. (25) we obtain that the 1-loop term is fully periodic with

$$\frac{\hbar}{2\bar{\eta}^R} = \frac{\hbar}{2\eta} - \frac{2}{\pi} \sin^2\left(\frac{\hbar}{2\eta}\right) \ln[\tau_1 v] \quad (78)$$

hence $\frac{\hbar}{2\eta} \rightarrow \sin(\frac{\hbar}{2\eta})$ in Eq. (73).

FIG. 4: RG flow of η .

We propose then that an RG consistent theory corresponds to

$$\frac{\hbar}{2\eta^R} = \frac{\hbar}{2\eta} - \frac{2}{\pi} \sin^2\left(\frac{\hbar}{2\eta}\right) \ln[\tau_1 v] + \frac{4}{\pi^2} \sin^3\left(\frac{\hbar}{2\eta}\right) \cos\left(\frac{\hbar}{2\eta}\right) \{\ln^2[\tau_1 v] + b_0(\eta) \ln[\tau_1 v]\} \quad (79)$$

Taking a sine of both sides it yields to order g^3 , with $b_0 = b_0(g=0)$,

$$g_R = g \mp g^2 \ln(v/\omega_c) + g^3 [\ln^2(v/\omega_c) + b_0 \ln(v/\omega_c)] \quad (80)$$

where \pm refers to $g=0$ with $\cos(\frac{\hbar}{2\eta}) = \pm 1$, leading to

$$\beta(g_R) = \frac{dg_R}{-d \ln v} = \pm g_R^2 - b_0 g_R^3 + O(g_R^4). \quad (81)$$

This RG equation is satisfied for both \pm fixed points as seen by substituting (80). We propose then that $g^R = 0$ are exact zeroes of the perturbation expansion and the additional requirement of an RG structure leads to the result (80).

Eq. (80) yields fixed points at $\frac{\hbar}{2\eta_n} = n\pi$ with $n = 1, 2, 3, \dots$ that are attractive at $\eta > \eta_n$ and repulsive at $\eta < \eta_n$, i.e. the flow of $\eta \neq \eta_n$ is always to smaller η . At these fixed points a Gaussian evaluation yields the correlation $\langle \cos \theta_t \cos \theta_0 \rangle \sim t^{-2n}$. We recall now a theorem for the lattice model²⁵ where the equilibrium action with mass related cutoff is replaced by an action on a lattice resulting in an XY model with long range interactions. The theorem states²⁵ that $\langle \cos \theta_t \cos \theta_0 \rangle \sim 1/t^2$; this result was also derived⁹ in first order in η . The range $\eta > \eta_1$ has an RG flow to η_1 and is therefore consistent with the theorem. The hypothesis of Gaussian fixed points corresponding to $n \geq 2$ is inconsistent with the theorem, i.e. $\langle \cos \theta_t \cos \theta_0 \rangle$ becomes a relevant operator at the $n \leq 2$ points rendering them unstable. Note that in the SEB problem $\cos \theta_t$ corresponds to a lead-dot voltage and its correlations determine the SET conductance^{11,13,20}, while in the ring problem it corresponds to fluctuations in the circular asymmetry.

For $\eta < \eta_1$ the system could have non-gaussian fixed points or a line of fixed points as hinted by the small η perturbation⁹. The equilibrium $K_1(\phi_x)$ was evaluated for small η and for $T \rightarrow 0$ has the form $K_1(\phi_x) \sim \delta(\phi_x - \frac{1}{2})/T$, i.e. the dissipation is concentrated at the single point $\phi_x = \frac{1}{2}$. This implies from Eq. (39) that $\eta^R \sim T$ and therefore vanishes at temperature $T = 0$. It is not clear, however, that $\eta = 0$ is a fixed point in the RG sense and if so what is its range of attraction. An $\eta = 0$ fixed point would imply the implausible result that the ring conductance diverges for small but finite η . We therefore expect that $\eta_1 \equiv \eta^R$ is the single fixed point in this system, as illustrated in Fig. 4.

V. DISCUSSION

The special value $\eta^R = \hbar/(2\pi)$ has a topological interpretation as a Thouless charge pump²⁷. Consider a slow change of ϕ_x by one unit with $\hbar \dot{\phi}_x = \eta^R \langle \dot{\theta} \rangle$. For the special value $\eta^R = \hbar/(2\pi)$ the total change in the position of the particle is $\int_t \langle \dot{\theta} \rangle dt = 2\pi$, i.e. the particle comes back to the same position on the ring and a unit charge has been transported. Such quantization has been shown for cases where the spectrum has a gap²⁷, though quantized charge transport was shown also in cases without a gap^{28,29}. The quantized η^R also results from arguing that there should be a unique frequency $\omega_E = \frac{2\pi}{\hbar} E = v$ as $E \rightarrow 0$ (see discussion below Eq. 38), as suggested by linear response.

We conclude from (45) that for $\eta > \eta_1 \equiv \eta^R$ the SEB satisfies the quantization

$$\int_0^1 \frac{C_0^2(N_0)}{C_g^2} R_q(N_0) dN_0 = \frac{\hbar}{e^2}. \quad (82)$$

In particular, when $\eta/\hbar \gtrsim 1$ we have⁶⁻⁹ from the known $M^*/m \sim e^{\pi\eta/\hbar}$ and from Eq. (6) $C_0/C_g = 1 + O(e^{-\pi\eta/\hbar})$. We expect R_q to be independent of N_0 at large η , hence

$$R_q = \frac{\hbar}{e^2} [1 + O(e^{-\pi\eta/\hbar})] \quad (83)$$

similar to the $N_c = 1$ case³.

The conductance of the ring can be defined by the voltage around the ring $2\pi E/e$ and the current $e\langle\dot{\theta}\rangle/2\pi$, hence we predict that the conductance for $\eta > \eta^R$ is

$$G_{ring} = \frac{e^2}{4\pi^2\eta^R} = \frac{e^2}{h}. \quad (84)$$

Finally, we consider the conditions for our proposed box experiment. By analogy with $E = \hbar\dot{\phi}_x$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 = Et$. This leads to a DC current into the Coulomb box whose dissipation is the average in Eq. (45). The field E should be sufficiently small so that g_R is sufficiently near the fixed point. For an initial $g \approx 1$ integration of $\partial g_R / \partial \ln E = g_R^2$ yields $g_R = 1 / \ln(\hbar\omega_c/E) \ll g$. E.g. for $g_R \lesssim 0.1$ and a typical $\hbar\omega_c \approx 1\text{meV}$ one needs $E/\hbar \lesssim 10^8\text{Hz}$. E/\hbar has frequency units, corresponding to 10^8 electrons/sec flowing into the box. We propose measuring the charge fluctuations (noise) $S_Q(\omega) = e^2 \langle \hat{N}_t \hat{N}_{t'} \rangle_\omega$ at a frequency, temperature and level spacings Δ such that $\Delta < \omega, T \ll 10^8\text{Hz}$, to yield the DC response of Eqs. (39,82). We predict then that the corresponding charge noise $S_Q(\omega) (\frac{2E_c}{e\hbar})^2 \frac{1}{\omega} = \frac{\hbar}{\eta^R} = 2\pi$ is quantized.

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Appendix A: Mapping the Coulomb box and the ring

The AES mapping has been extensively used, yet we find it useful to reproduce it since the relation between correlation functions has received less attention.

The Coulomb box action corresponding to the Hamiltonian (40) is

$$\begin{aligned} -i\hbar S &= \int_t \left\{ \sum_\alpha d_{\alpha,i}^\dagger (i\hbar\partial_t - \epsilon_\alpha) d_{\alpha,i} - E_c (\hat{N} - N_0)^2 \right\} - i\hbar S_{lead} - i\hbar S_{tun} \\ -i\hbar S_{lead} &= \int_t \sum_k a_{k,i}^\dagger (i\hbar\partial_t - \epsilon_k) a_{k,i} \\ -i\hbar S_{tun} &= \int_t \sum_{k,\alpha} t_{k,\alpha,i} a_{k,i}^\dagger d_{\alpha,i} + h.c. \end{aligned} \quad (A1)$$

with the partition $Z = e^{-S}$. Adding a variable $\dot{\theta}_t$ to the path integral yields

$$\begin{aligned} -i\hbar S &= \int_t \left\{ E_c [\hat{N} - N_0 - \frac{\hbar}{2E_c} \dot{\theta}_t]^2 + \sum_\alpha d_{\alpha,i}^\dagger (i\hbar\partial_t - \epsilon_\alpha) d_{\alpha,i} - E_c (\hat{N} - N_0)^2 \right\} - i\hbar S_{lead} - i\hbar S_{tun} \\ &= \int_t \left\{ \sum_\alpha d_{\alpha,i}^\dagger (i\hbar\partial_t - \epsilon_\alpha - \hbar\dot{\theta}_t) d_{\alpha,i} + \frac{1}{4E_c} [\hbar\dot{\theta}_t + 2E_c N_0]^2 \right\} - i\hbar S_{lead} - i\hbar S_{tun} \end{aligned} \quad (A2)$$

Now define $d_\alpha = e^{-i\theta_t} \tilde{d}_\alpha$

$$-i\hbar S = \int_t \left\{ \sum_\alpha \tilde{d}_{\alpha,i}^\dagger (i\hbar\partial_t - \epsilon_\alpha) \tilde{d}_{\alpha,i} + \frac{\hbar^2}{4E_c} \dot{\theta}_t^2 + \dot{\theta}_t N_0 + \sum_{k,\alpha,i} [t_{k,\alpha,i} a_{k,i}^\dagger \tilde{d}_{\alpha,i} e^{i\theta_t} + h.c.] \right\} - i\hbar S_{lead} \quad (A3)$$

The ring action in terms of θ_t is derived by integrating out the fermions \tilde{d}_α and a_k . Define time ordered Greens' functions on the dot $G_{0\alpha,i}(\omega) = \frac{1}{\omega - \epsilon_{\alpha,i} + i\text{sign}\omega 0^+}$ and on the lead $G_{0k,i}(\omega) = \frac{1}{\omega - \epsilon_{k,i} + i\text{sign}\omega 0^+}$. In matrix notation

$$\hat{G}_i^{-1}(t, t') = \begin{pmatrix} G_{0\alpha,i}^{-1}(t, t') & 0 \\ 0 & G_{0k,i}^{-1}(t, t') \end{pmatrix} + \begin{pmatrix} 0 & t_{k,\alpha,i} e^{i\theta_t} \\ t_{k,\alpha,i}^* e^{-i\theta_t} & 0 \end{pmatrix} \delta(t - t') \equiv \hat{G}_{0i}^{-1} + \hat{T}_i \quad (A4)$$

The trace over fermions, using $\det(iG) = e^{Tr \ln iG}$, yields

$$S_{eff} = - \sum_i Tr \ln i\hat{G}_i^{-1}(t, t) = - \sum_i Tr \ln \left\{ i\hat{G}_{0i}^{-1}(t, t') [\delta(t - t') + \hat{G}_{0i}(t', t) \hat{T}_i(t)] \right\} \quad (A5)$$

Expanding in \hat{T} , the 0th order is θ_t independent, the 1st order vanishes, hence to 2nd order

$$S_{eff} = -\frac{1}{2} \sum_i Tr \{ \hat{G}_0 \hat{T} \hat{G}_0 \hat{T} \} = -\frac{1}{2} \sum_i \int_{t, t'} G_{0\alpha, i}(t, t') G_{0k, i}(t', t) |t_{k, \alpha, i}|^2 e^{i\theta_t - i\theta_{t'}} + h.c. \quad (A6)$$

For completeness we derive the Matsubara effective action using $\sum_\alpha G_{\alpha, i}(\tau) = T \sum_n G(\omega_n) e^{i\omega_n \tau}$ with fermionic $\omega_n = \pi T(2n + 1)$,

$$G(\omega_n) = \int_\epsilon \frac{\rho_{dot}(\epsilon)}{i\omega_n - \epsilon} = \int_0^\infty \rho_{dot}(\epsilon) \left[\frac{1}{i\omega_n - \epsilon} + \frac{1}{i\omega_n + \epsilon} \right] = \int_0^\infty \rho_{dot}(\epsilon) \frac{-2i\omega_n}{\omega_n^2 + \epsilon^2} = -i\pi \rho_{dot}(0) \text{sgn}(\omega_n)$$

$$\sum_\alpha G_{0\alpha, i}(\tau) = 2\pi \rho_{dot}(0) \sum_{n>0} \sin(\omega_n \tau) = \rho_{dot}(0) \frac{\pi T}{\sin(\pi T \tau)} \quad (A7)$$

where $\rho_{dot}(\epsilon)$ is the dot density of states, assumed symmetric, and eventually constant. With the lead density of states $\rho_{lead}(\epsilon)$, and assuming a constant $t_{k, \alpha, i}$

$$S_{eff} = -\frac{1}{2} |t|^2 N_c \rho_{dot}(0) \rho_{lead}(0) \int \int \frac{\pi^2 T^2}{\sin^2[\pi T(\tau - \tau')]} \cos[\theta(\tau) - \theta(\tau')] \quad (A8)$$

where $N_c = \sum_i$ is the number of channels. This is the well known equilibrium ring system with a bosonic CL environment⁶⁻⁹ where $\eta = \frac{1}{2} \pi |t|^2 N_c \rho_{dot}(0) \rho_{lead}(0)$ and $m = 1/2E_c$. The expansion in \hat{T} is justified for $|t|^2 \rightarrow 0$, however with $N_c \rightarrow \infty$ any value of η can be generated. In fact N_c can be fairly small and yet reproduce the $N_c \rightarrow \infty$ case, except at exponentially small temperatures³². A similar derivation holds for the Keldysh action leading to the form (21).

We proceed now to map observables of the Coloumb box to those of the ring problem. Since the action (A3) has a term $+\dot{\theta} N_0$ we identify $N_0 = -\phi_x$ where ϕ_x is the flux through the ring (in units of the quantum flux). Hence

$$\begin{aligned} \hbar \langle \dot{\theta} \rangle &= \int_\theta \hbar \dot{\theta} e^{-\frac{i}{\hbar} \int E_c (\hat{N} - N_0 - \frac{\hbar}{2E_c} \dot{\theta})^2 + \text{fermion terms}} \\ &= \int_\theta (\hbar \dot{\theta} + 2E_c \hat{N} - 2E_c N_0) e^{-\frac{i}{\hbar} \int \frac{\hbar^2}{4E_c} \dot{\theta}^2 + \text{fermion terms}} = 2E_c [\langle \hat{N} \rangle - N_0] \end{aligned} \quad (A9)$$

In particular, without interaction, $t_{k\alpha} = 0$, the charge has no fluctuations $\langle \hat{N} \rangle = 0$ (for $|N_0| < \frac{1}{2}$) so that $\hbar \langle \dot{\theta} \rangle = -2E_c N_0 = 2E_c \phi_x$.

Consider next the time ordered \mathcal{T} correlations (the following is the same for $\langle \dot{\theta}_t^+ \dot{\theta}_{t'}^+ \rangle, \langle \dot{\theta}_t^+ \dot{\theta}_{t'}^- \rangle$ with \pm Keldysh contours),

$$\begin{aligned} \hbar^2 \mathcal{T} \langle \dot{\theta}_t \dot{\theta}_{t'} \rangle &= \int_{\dot{\theta}} \hbar^2 \dot{\theta}_t \dot{\theta}_{t'} e^{-\frac{i}{\hbar} \int E_c (\hat{N} - N_0 - \frac{\hbar}{2E_c} \dot{\theta})^2 + \text{fermion terms}} \\ &= \int_{\dot{\theta}} (\hbar \dot{\theta}_t + 2E_c \hat{N}_t - 2E_c N_0) (\hbar \dot{\theta}_{t'} + 2E_c \hat{N}_{t'} - 2E_c N_0) e^{-\frac{i}{\hbar} \int \frac{\hbar^2}{4E_c} \dot{\theta}^2 + \text{fermion terms}} \\ &= \hbar^2 \mathcal{T} \langle \dot{\theta}_t \dot{\theta}_{t'} \rangle_0 + 4E_c^2 \mathcal{T} \langle (\hat{N}_t - N_0) (\hat{N}_{t'} - N_0) \rangle \end{aligned} \quad (A10)$$

To obtain the retarded response,

$$-i\mathcal{D}_{t, t'}^R = \theta(t - t') \langle [A_t, B_{t'}] \rangle = \theta(t - t') \langle A_t B_{t'} - B_{t'} A_t \rangle = T \langle A_t^+ B_{t'}^+ \rangle - \langle B_{t'}^- A_t^+ \rangle \quad (A11)$$

where \pm are Keldysh contour indices, so that A^+ is earlier than B^- .

Define the response $K_{t, t'}$ of the Coulomb box, as well as the response of ring problem $\tilde{K}_{t, t'}$ in the form (displayed here with operators whose $\langle A_t \rangle = 0$ to allow relation with time ordering),

$$\begin{aligned} \tilde{K}_{t, t'} &= +i\theta(t - t') \langle [(\dot{\theta}_t - \langle \dot{\theta} \rangle), (\dot{\theta}_{t'} - \langle \dot{\theta} \rangle)] \rangle \\ K_{t, t'} &= +i\theta(t - t') \langle [(\hat{N}_t - \langle \hat{N} \rangle), (\hat{N}_{t'} - \langle \hat{N} \rangle)] \rangle \end{aligned} \quad (A12)$$

From Eq. (A10) we have

$$\begin{aligned} & \hbar^2 \mathcal{T} \langle (\dot{\theta}_t - \langle \dot{\theta} \rangle) (\dot{\theta}_{t'} - \langle \dot{\theta} \rangle) \rangle + \hbar^2 \langle \dot{\theta} \rangle^2 = \\ & \hbar^2 \mathcal{T} \langle \dot{\theta}_t \dot{\theta}_{t'} \rangle_0 + 4E_c^2 \mathcal{T} \langle (\hat{N}_t - \langle \hat{N} \rangle) (\hat{N}_{t'} - \langle \hat{N} \rangle) \rangle + 4E_c^2 (\langle \hat{N} \rangle^2 - 2N_0 \langle \hat{N} \rangle + N_0^2) \end{aligned} \quad (\text{A13})$$

Now using (A9) and that the relation (A10) holds for both terms in (A11), a relation between these response functions is obtained

$$\hbar^2 \tilde{K}_{t,t'} = -2E_c \hbar \delta(t - t') + 4E_c^2 K_{t,t'} \quad (\text{A14})$$

which is reproduced as Eq. (40). This relation is consistent with results in Ref. 21.

Appendix B: Semiclassical case: 1st and 2nd order

1. 1st order term

First order perturbation of the Green's function

$$\begin{aligned} R_{t,t'}^{(1)} &= -i \frac{1}{2} \int_{t_1, t_2} B_{t_1, t_2} \left\langle \hat{\theta}_{t_1} \hat{\theta}_{t_2} \cos(\theta_{t_1} - \theta_{t_2}) \hat{\theta}_{t'} \theta_t \right\rangle_{S_0} = \\ & \frac{-i}{4} \int_{t_1, t_2} B_{t_1, t_2} \sum_{\sigma=\pm} \partial_{\alpha_{i=1,2,3,4}} \text{Exp} \left[i\alpha_1 \hat{\theta}_{t_1} + i\alpha_2 \hat{\theta}_{t_2} + i\sigma \theta_{t_1} - i\sigma \theta_{t_2} + i\alpha_3 \hat{\theta}_{t'} + i\alpha_4 \theta_t \right] \Big|_{\alpha_i=0} \end{aligned} \quad (\text{B1})$$

An Averaging with Gaussian weight

$$\begin{aligned} & \left\langle e^{i\theta_{t_1} + i\theta_{t_2} + \dots + i\hat{\theta}_{t_1} + i\hat{\theta}_{t_2} + \dots} \right\rangle = e^{i\langle \theta_{t_1} + \theta_{t_2} + \dots \rangle} e^{-\langle (\theta_{t_1} + \theta_{t_2} + \dots) (\hat{\theta}_{t_1} + \hat{\theta}_{t_2} + \dots) \rangle} = \\ & e^{ivt_1 + ivt_2 + \dots} e^{iR_{t_1, t_2} + iR_{t_2, t_1} + \dots} \end{aligned} \quad (\text{B2})$$

The retarded function

$$\begin{aligned} R_{t,t'}^{(1)} &= \\ & \frac{1}{4i} \int_{t_1, t_2} \sum_{\sigma=\pm} \partial_{\alpha_i} B_{t_1, t_2} e^{i\alpha_1(-\sigma R_{t_2, t_1} + \alpha_4 R_{t, t_1}) + i\alpha_2(\sigma R_{t_1, t_2} - \alpha_4 R_{t, t_1}) + i\alpha_3(\sigma R_{t_1, t'} - \sigma R_{t_2, t'} + \alpha_4 R_{t, t_1})} e^{i\sigma v(t_1 - t_2)} = \\ & \frac{1}{4} \int_{t_1, t_2} \sum_{\sigma=\pm} \partial_{\alpha_4} B_{t_1, t_2} (\sigma R_{t_2, t_1} - \alpha_4 R_{t, t_1}) (\sigma R_{t_1, t_2} + \alpha_4 R_{t, t_1}) (\sigma R_{t_1, t'} - \sigma R_{t_2, t'} + \alpha_4 R_{t, t_1}) e^{i\sigma v(t_1 - t_2)} = \\ & - \int_{t_1, t_2} B_{t_1, t_2} \cos v(t_1 - t_2) R_{t, t_1} R_{t_1, t_2} (R_{t_1, t'} - R_{t_2, t'}) \end{aligned} \quad (\text{B3})$$

In the last expression we use $R_t R_{-t} = 0$.

2. 2nd order term

Using the same procedure for the second order

$$\begin{aligned} R_{t,t'}^{(2)} &= \frac{i}{2} \left\langle \hat{\theta}_{t'} \theta_t (S_{int})^2 \right\rangle = \\ & - \frac{i}{8} \int_{t_1, t_2, t_3, t_4} B_{t_1, t_2} B_{t_3, t_4} \left\langle \hat{\theta}_{t_1} \hat{\theta}_{t_2} \cos(\theta_{t_1} - \theta_{t_2}) \hat{\theta}_{t_3} \hat{\theta}_{t_4} \cos(\theta_{t_3} - \theta_{t_4}) \hat{\theta}_{t'} \theta_t \right\rangle = \frac{1}{2^5 i} \int_{t_1, \dots, t_4} B_{t_1, t_2} B_{t_3, t_4} \\ & \times \sum_{\sigma_1, \sigma_2=\pm} \partial_{\alpha_{i=1, \dots, 6}} \left\langle e^{i\alpha_1 \hat{\theta}_{t_1} + i\alpha_2 \hat{\theta}_{t_2} + i\alpha_3 \hat{\theta}_{t_3} + i\alpha_4 \hat{\theta}_{t_4} + i\sigma_1 \theta_{t_1} - i\sigma_1 \theta_{t_2} + i\sigma_2 \theta_{t_3} - i\sigma_2 \theta_{t_4} + i\alpha_3 \hat{\theta}_{t'} + i\alpha_4 \theta_t} \right\rangle \Big|_{\alpha_i=0} \end{aligned} \quad (\text{B4})$$

using the symmetry between $\sigma_1 \leftrightarrow -\sigma_1$ and $t_1 \leftrightarrow t_2$ and similarly for t_3, t_4

$$\begin{aligned} R_{t,t'}^{(2)} &= \frac{1}{8} \int_{t_1, t_2, t_3, t_4} B_{t_1, t_2} B_{t_3, t_4} e^{iv(t_1 - t_2) - iv(t_3 - t_4)} \partial_{\alpha_6} [-R_{t_2, t_1} + R_{t_3, t_1} - R_{t_4, t_1} + \alpha_6 R_{t, t_1}] \\ & [R_{t_1, t_2} + R_{t_3, t_2} - R_{t_4, t_2} + \alpha_6 R_{t, t_2}] [R_{t_1, t_3} - R_{t_2, t_3} - R_{t_4, t_3} + \alpha_6 R_{t, t_3}] \\ & [R_{t_1, t_4} - R_{t_2, t_4} + R_{t_3, t_4} + \alpha_6 R_{t, t_4}] [R_{t_1, t'} - R_{t_2, t'} + R_{t_3, t'} - R_{t_4, t'} + \alpha_6 R_{t, t'}] \end{aligned} \quad (\text{B5})$$

the choice $t_1 > t_2, t_3, t_4$, only R_{t,t_1} remains. R_τ is real, we separate the exponent to two sinus and two cosine terms as follow

$$R_{t,t'}^{(2)} = \frac{1}{8} \int_{t_1, t_2, t_3, t_4} B_{t_1, t_2} B_{t_3, t_4} (\cos v(t_1 - t_2) \cos v(t_3 - t_4) - \sin v(t_1 - t_2) \sin v(t_3 - t_4)) R_{t, t_1} [R_{t_1, t_2} + R_{t_3, t_2} - R_{t_4, t_2}] [R_{t_1, t_3} - R_{t_2, t_3} - R_{t_4, t_3}] [R_{t_1, t_4} - R_{t_2, t_4} + R_{t_3, t_4}] [R_{t_1, t'} - R_{t_2, t'} + R_{t_3, t'} - R_{t_4, t'}] \quad (\text{B6})$$

This long multiplicity of R_t terms is now separated to 8 different terms. For the terms with the cosine we calculate explicitly 3 terms, which we label by a to c . Term 'a':

$$R_{t,t'}^a = \frac{1}{2} \int_{t_1, t_2, t_3, t_4} B_{t_1, t_2} \cos v(t_1 - t_2) \times R_{t, t_1} R_{t_1, t_2} (R_{t_1, t'} - R_{t_2, t'}) B_{t_3, t_4} \cos v(t_3 - t_4) (R_{t_1, t_3} - R_{t_2, t_3}) (R_{t_1, t_4} - R_{t_2, t_4}) = \frac{1}{2} \int_{t_1, t_2} B_{t_1, t_2} \cos v(t_1 - t_2) R_{t, t_1} R_{t_1, t_2} (R_{t_1, t'} - R_{t_2, t'}) \tilde{C}_{t_1, t_2} \quad (\text{B7})$$

This term in ω space

$$R_\omega^a = -\frac{1}{2} R_\omega^2 \int_t R_t B_t \cos vt (e^{i\omega t} - 1) \tilde{C}_t$$

with $\tilde{C}_t = 2(C_{t=0}^{(1)} - C_t^{(1)})$. Similarly we choose two different terms 'b' and 'c' and write them directly in ω space

$$R_\omega^b = R_\omega^2 \int_t R_t^{(1)} B_t \cos vt (e^{i\omega t} - 1) \quad (\text{B8})$$

$$R_\omega^c = R_\omega^3 \left[\int_t R_t B_t \cos vt (e^{i\omega t} - 1) \right]^2 = R_\omega^{-1} (R_\omega^{(1)})^2 \quad (\text{B9})$$

note the $R_t^{(1)}$ in the expression R^b is the first order result of the retarded green function. R_ω^c is the reducible term containing multiplication of $R_\omega^{(1)}$. Renormalized η for small v is

$$\begin{aligned} \frac{1}{\eta_2^a} &= \frac{1}{2} \frac{1}{\eta^2} \int_t R_t B_t \tilde{C}(t) t = \frac{\hbar}{\pi \eta^3} \int_t R_t B_t t (\log t + \gamma + \mathcal{O}(v) + \mathcal{O}(1/t)) = -\frac{\hbar^2}{2\pi^2 \eta^3} \log^2 v + \mathcal{O}(v) \\ \frac{1}{\eta_2^b} &= -\frac{\hbar}{\pi \eta^2} \int_t R_t^{(1)} B_t t = -\frac{\hbar}{\pi \eta^3} \int_t R_t B_t t (\log t + \gamma + 1 + \mathcal{O}(v) + \mathcal{O}(1/t)) = \\ &\quad \frac{\hbar^2}{2\pi^2 \eta^3} \log^2 v - \frac{\hbar^2}{2\pi^2 \eta^3} \log v + \mathcal{O}(v) \\ \frac{1}{\eta_2^c} &= \frac{1}{\eta^3} \left[\int_t R_t B_t t \right]^2 = \frac{\hbar^2}{2\pi^2 \eta^3} [\log v + \mathcal{O}(v)]^2 = \frac{\hbar^2}{2\pi^2 \eta^3} \log^2 v + \mathcal{O}(v) \end{aligned} \quad (\text{B10})$$

The terms containing the sine in Eq. (B6), are in general of order $\mathcal{O}(v)$, however we have identify the following term which, depending on the order of limits, may contribute a term logarithmic in v for small v .

$$R_\omega^d = -R_\omega^2 \int_{t_1, t_2} R_{t_1} R_{t_2} B_{t_1} B_{t_2} \sin vt_1 \sin vt_2 (1 - e^{i\omega t_1}) \int_{t_3} (R_{t_1+t_3} - R_{t_3}) \quad (\text{B11})$$

We label the dissipation parameter from this term by $\delta(\frac{1}{\eta_R^d}) = \lim_{\omega \rightarrow 0} (-i\omega) R_\omega^d$ and found the logarithmic prefactor in Eq. (56), where we use for $t_1 > 0$

$$\int_{t_3} (R_{t_1+t_3} - R_{t_3}) = \frac{1}{\eta} \int_{-t_1}^0 \left(1 - e^{-(t_1+t_3) \frac{\eta}{m}} \right) + \frac{1}{\eta} \int_0^\infty \left(e^{-t_3 \frac{\eta}{m}} - e^{-(t_1+t_3) \frac{\eta}{m}} \right) = \frac{t_1}{\eta} \quad (\text{B12})$$

Appendix C: Quantum case: 1st order, more details

Let us give the detailed calculation of the first order correction in the case of a mass only cutoff, i.e. $\tau_0 = 0$. Taking the derivative of Eq. (63) in the text we have:

$$\begin{aligned}
\partial_v \delta E^{(1)} &= -\frac{2}{\hbar} \int_{\tau>0} \tau B(\tau) \sin\left(\frac{\hbar}{2} R(\tau)\right) \cos(v\tau) = \frac{2\eta}{\pi} \int_{\tau>0} \frac{d\tau}{\tau} \sin\left(\frac{\hbar}{2\eta}(1 - e^{-\frac{\eta}{m}\tau})\right) \cos(v\tau) \\
&= \frac{2\eta}{\pi} \left[\sin\left(\frac{\hbar}{2\eta}\right) \int_{\tau>0} \frac{d\tau}{\tau} (1 - e^{-\frac{\eta}{m}\tau}) \cos(v\tau) \right. \\
&\quad \left. - \int_{\tau>0} \frac{d\tau}{\tau} \left[\sin\left(\frac{\hbar}{2\eta}(1 - e^{-\frac{\eta}{m}\tau})\right) - \sin\left(\frac{\hbar}{2\eta}\right)(1 - e^{-\frac{\eta}{m}\tau}) \right] \cos(v\tau) \right] \\
&= \frac{2\eta}{\pi} \left[\sin\left(\frac{\hbar}{2\eta}\right) \ln\left(\frac{\eta}{mv}\right) + f\left(\frac{\hbar}{2\eta}\right) + O(1/v) \right]
\end{aligned} \tag{C1}$$

since the first integral can be computed exactly and in the second one can set $v = 0$ to get the constant piece. This determines the constant $C = f(\frac{\hbar}{2\eta})$ given in the text in Eq. (64), where the function $f(x)$ is defined as:

$$\begin{aligned}
f(x) &= \int_0^{+\infty} \frac{dt}{t} [\sin(x(1 - e^{-t})) - \sin(x)(1 - e^{-t})] \\
&= -\int_0^1 \frac{dz}{(1-z)\ln(1-z)} (\sin(xz) - z \sin x) = \frac{1}{6} x^3 \log\left(\frac{8}{3}\right) + O(x^5)
\end{aligned} \tag{C2}$$

and is a nicely convergent integral, where one can rescale t freely. Although it is not periodic in x , upon plotting it one notes that it seems to become almost periodic a large x .

Appendix D: Quantum case: 2nd order for $\tau_1 \rightarrow 0$

Since $\sin(\frac{1}{2}\hbar R_{t_1, t_2})$ is a retarded function, we use for $R_t = \Theta(t)e^{-\delta t}$

$$\begin{aligned}
\sin\left(\frac{1}{2}\hbar R_{t_1, t_2}\right) &\rightarrow \sin\left(\frac{\hbar}{2\eta}\right) e^{-\delta(t_1 - t_2)} \\
\sin\left[\frac{1}{2}\hbar(R_{t_1, t_3} + R_{t_2, t_3})\right] &\rightarrow \sin\left(\frac{\hbar}{\eta}\right) e^{-\delta(t_1 - t_3) - \delta(t_2 - t_3)} \\
\sin\left[\frac{1}{2}\hbar(R_{t_1, t_4} - R_{t_3, t_4} + R_{t_2, t_4})\right] &\rightarrow \sin\left(\frac{\hbar}{2\eta}\right) e^{-\delta(t_1 - t_4) - \delta(t_3 - t_4) - \delta(t_2 - t_4)}
\end{aligned} \tag{D1}$$

e.g. Fourier of $t_1 - t_3$ and $t_2 - t_3$ should have $1/(\omega_1 + i\delta)(\omega_2 + i\delta)$. define the variables

$$\begin{aligned}
t'_2 &= t_2 - t_1, & t'_3 &= t_3 - t_2, & t'_4 &= t_4 - t_3 \\
\Rightarrow t_2 &= t'_2 + t_1, & t_3 &= t'_3 + t'_2 + t_1, & t_4 &= t'_4 + t'_3 + t'_2 + t_1
\end{aligned} \tag{D2}$$

These variables are more convenient since their range is independent $-\infty < t'_2, t'_3, t'_4 < 0$. The product of all convergence factors is then $e^{\delta(3t'_2 + 4t'_3 + 3t'_4)}$, with 3,4,3 factors unimportant since $\delta \rightarrow 0$. Hence

$$\begin{aligned}
\delta E^{(2)} &= \frac{4}{\hbar^3} \sin^2\left(\frac{\hbar}{2\eta}\right) \sin\left(\frac{\hbar}{\eta}\right) \int_{\omega_1, \omega_2} B(\omega_1) B(\omega_2) \sum_{\sigma=\pm} \frac{\sigma}{2i} \int_A e^{i\sigma v(-2t'_3 - t'_4 - t'_2)} \\
&\times [e^{i\omega_1(t'_3 + t'_2) + i\omega_2(t'_4 + t'_3)} + e^{i\omega_1(t'_4 + t'_3 + t'_2) + i\omega_2 t'_3}] e^{\delta(t'_2 + t'_3 + t'_4)} = \\
&\frac{4}{\hbar^3} \sin^2\left(\frac{\hbar}{2\eta}\right) \sin\left(\frac{\hbar}{\eta}\right) \int_{\omega_1, \omega_2} B(\omega_1) B(\omega_2) \sum_{\sigma=\pm} \frac{\sigma}{2i} \times \\
&\left[\frac{1}{-i\sigma v + i\omega_2 + \delta} + \frac{1}{-i\sigma v + i\omega_1 + \delta} \right] \frac{1}{(-2i\sigma v + i\omega_1 + i\omega_2 + \delta)(-i\sigma v + i\omega_1 + \delta)} = \\
&\frac{4}{\hbar^3} \sin^2\left(\frac{\hbar}{2\eta}\right) \sin\left(\frac{\hbar}{\eta}\right) \sum_{\sigma} \frac{\sigma}{2} (\hbar\eta)^2 \int \frac{d\omega_1}{2\pi} \frac{1}{(\omega_1 - \sigma v - i\delta)^2} \frac{|\omega_1|}{1 + \omega_1^2 \tau_0^2} \int \frac{d\omega_2}{2\pi} \frac{1}{\omega_2 - \sigma v - i\delta} \frac{|\omega_2|}{1 + \omega_2^2 \tau_0^2}
\end{aligned} \tag{D3}$$

with the integral over ω_2

$$\begin{aligned} \int_0^\infty d\omega_2 \left[\frac{1}{\omega_2 - \sigma v - i\delta} - \frac{1}{-\omega_2 - \sigma v - i\delta} \right] \frac{\omega_2}{1 + \omega_2^2 \tau_0^2} &= 2\sigma v \int_0^\infty d\omega_2 \frac{\omega_2}{(\omega_2^2 - v^2)(1 + \omega_2^2 \tau_0^2)} \\ &= -\sigma v \ln(v\tau_0) + O(v^3 \tau_0^2 \ln(v\tau_0)) \end{aligned} \quad (\text{D4})$$

and over ω_1

$$\begin{aligned} \int_0^\infty d\omega_1 \left[\frac{1}{(\omega_1 - \sigma v - i\delta)^2} + \frac{1}{(-\omega_1 - \sigma v - i\delta)^2} \right] \frac{\omega_1}{1 + \omega_1^2 \tau_0^2} \\ = 2 \int_0^\infty d\omega_1 \left[\frac{\omega_1}{\omega_1^2 - v^2} + \frac{2v^2 \omega_1}{(\omega_1 - \sigma v - i\delta)^2 (\omega_1 + \sigma v + i\delta)^2} \right] = -2 \ln(v\tau_0) - 2 \end{aligned} \quad (\text{D5})$$

where in the last integral $\tau_0 \rightarrow 0$ can be taken. Substituting (D4,D5) in (D3) leads to the result Eq. (70).

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